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SYLOW SUBGROUPS IN THE GROUP OF ISOMORPHISMS OF PRIME POWER ABELIAN GROUPS.

By Harry Albert Bender.

1. Introduction.

The study of groups of isomorphisms is a comparatively recent one. C. Jordan in "Traité des substitutions" (1870), p. 56, introduced the term isomorphism in the following manner: A group G' is said to be isomorphic to another group G, if one can establish between their substitutions a correspondence such: 1° that each substitution of G corresponds to only one substitution of G', and each substitution of G' to one or more substitutions of G; 2° that the product of any two substitutions of G corresponds to the product of their respective correspondents. If G and G' are simply isomorphic they are identical as abstract groups.

The far more important concept of isomorphisms was introduced by O. Hölder and E. H. Moore in 1893–94, viz., the representation of the different automorphisms of a group by the corresponding substitutions on its operators. The totality of these substitutions constitute a group known as the group of isomorphisms. The group properties of these groups were first studied by O. Hölder and E. H. Moore, but the more important results have been contributed by G. A. Miller and W. Burnside.

In this article we shall be wholly interested in the groups of isomorphisms I of prime power abelian groups. Many of the researches on isomorphisms of prime power abelian groups have been on the representation of the group of isomorphisms as a substitution group.

The following are some of the more important abstract properties of the groups of isomorphisms that have been obtained. The number of invariant operators in the I of G, the necessary and sufficient condition that the group of isomorphisms be abelian, the necessary and sufficient condition that two operators in the I of G be commutative, and if G has for its order a power of a prime number, the method of constructing a principal series, the order of a Sylow subgroup S whose order is a power of p in the I of G, the necessary and sufficient condition that there be but one such Sylow subgroup in I, and the necessary and sufficient condition that this Sylow subgroup S be abelian. In case G is of type $(1, 1, 1, \cdots)$ the number of Sylow subgroups, whose orders are a power of p in the I of G, is known, as well as many properties of special groups.

In this article we shall establish several general theorems and relations which will aid in determining more completely the properties of the group of isomorphisms, especially with reference to Sylow subgroups whose orders are a power of p in the I of G. We shall give the total number of such Sylow subgroups in I and some of their invariant subgroups, including their commutator subgroup and the subgroup common to all of them.

The determination of all the possible orders of operators in the group of isomorphisms seems to present a very difficult problem.

In the following we have determined the order of the group of isomorphisms, and have determined a necessary and sufficient condition that the group of isomorphisms has for its order a power of a prime number. We have shown that there are but p operators of G invariant under a given Sylow subgroup S of I, and have determined the total number of p-isomorphisms in the I of G.

2. CHARACTERISTIC SUBGROUPS OF G.

If G is an abelian group of order p^m, p being a prime number, and if P represents a subgroup, whose order is a power of p, in the group of isomorphisms I of G, then G involves a series of subgroups of orders $p, p^2, p^3, p^4, \cdots, p^{m-1}$ respectively, such that each of these subgroups except the last is included in the one which follows it, and each is invariant under P. Now consider all the possible automorphisms of G in which every operator corresponds to itself multiplied either on the left or on the right by some operator in the preceding subgroup. Each of these automorphisms of G corresponds to an operator, whose order is a power of p, in the group of isomorphisms of G. W. Burnside calls the automorphisms of G, which give rise to the operators of P, p-isomorphisms.*

It is known that a necessary and sufficient condition that an operator t in the group of isomorphisms of a group G of order p^m has for its order a power of p is that t transforms every operator in the series of subgroups

$$G_0, G_1, G_2, G_3, \cdots, G_m \quad (G_m = G)$$

of orders $1, p, p^2, p^3, p^4, \cdots, p^m$ respectively, into itself multiplied by an operator in the preceding subgroup.

Since every non-cyclic abelian group of order p^m is the direct product of independent cyclic groups, a set of independent generators of an abelian group is commonly used with the restricted meaning that the group generated by any number of these generators has only the identity in common with the group generated by the remaining operators of the set. In what follows we shall use a set of independent generators in this restricted sense,

* W. Burnside, *Proceeding of the London Mathematical Soc.*, Ser. 2, Vol. 11, p. 225.

and suppose such a set of independent generators of G to be of orders $p^{\alpha_1}, p^{\alpha_2}, p^{\alpha_3}, \cdots, p^{\alpha_n}$ $(\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_n > 0)$, and that the number of the independent generators of these orders is $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ respectively.

Consider the subgroups

$$H_1, H_2, H_3, H_4, \cdots, H_n, \tag{1}$$

which are respectively generated by the independent generators of the same order. Each subgroup has only the identity in common with any other subgroup, and the orders of these subgroups are $p^{\lambda_1\alpha_1}, p^{\lambda_2\alpha_2}, p^{\lambda_3\alpha_3}, \cdots, p^{\lambda_n\alpha_n}$ respectively. If s is any operator of order p^δ in H_i $(\delta \leqq \alpha_i, i \leqq n)$, then under the group of isomorphisms s is conjugate with the operators obtained by multiplying all the operators of order p^δ in H_i by the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of H_β whose orders do not exceed $p^{\alpha\beta-\alpha_i+\delta}$ $(\alpha_\beta - \alpha_i + \delta > 0)$, where β takes successively the values $i+1$, $i+2$, $i+3$, $\cdots$, n.* When $\delta = 1$, $\alpha_\beta - \alpha_i + \delta \leqq 0$, and hence the operators of order p in $\{H_1, H_2, H_3, \cdots, H_i\}$ not in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$ form a complete set of conjugate operators under the I of G $(i = 1, 2, 3, \cdots, n)$. In the special case when all the invariants are equal, then the operators of the same order form a complete set of conjugate operators under I.

A subgroup of G, which corresponds to itself in every possible automorphism of the group, is called a characteristic subgroup of G. This term was first used by Frobenius in Berliner Sitzungsberichte, 1895, p. 183.

All the possible characteristic subgroups (besides the identity) have a certain characteristic subgroup in common. This is called the fundamental characteristic subgroup of G, and is the only one in which all the operators, besides the identity, are conjugate under I. All these operators must be of order p, and must be the same power of independent generators, and hence must be the subgroup composed of all the operators of order p generated by the independent generators of highest order in a given set.

It has been shown that if s is any operator of order p^δ in H_i $(\delta \leqq \alpha_i, i \leqq n)$, then under I s is conjugate with the operators obtained by multiplying all the operators of order p^δ in H_i by the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of H_β whose orders do not exceed $p^{\alpha\beta-(\alpha_i-\delta)}$ $(\alpha_\beta - \alpha_i + \delta > 0)$, where β takes successively the values $i+1, i+2, \cdots, n$. These operators of order p^δ are such that they generate a characteristic subgroup which is the smallest characteristic subgroup that contains all the operators of order p^δ in H_i. If we multiply all the operators of order p^σ in H_i into the group generated by all the operators whose orders do not exceed p^δ in

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 27, p. 18.

$\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators whose orders are less than p^δ in $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$, then these operators generate a second characteristic subgroup such that in general there exists in this second characteristic subgroup many characteristic subgroups each of which contains the first as a constituent. We may determine these characteristic subgroups as follows. *A necessary and sufficient condition that a subgroup of G which contains the operators of order p^δ in H_i ($\delta \leqq \alpha_i$, $i \leqq n$), but contains no operator of higher order, be characteristic, is that it be generated by operators obtained by multiplying all the operators of order p^δ in H_i into the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators in H_β whose orders do not exceed $p^{\alpha'_\beta}$, where $\alpha'_{\beta-1} \geqq \alpha'_\beta \geqq \alpha_\beta - \alpha_{\beta-1} + \alpha'_{\beta-1}$ ($\alpha'_\beta \leqq \alpha_\beta$) as β takes successively the values $i+1, i+2, \cdots, n$ ($\alpha'_i \equiv \delta$).*

This follows at once, for if an operator s of order p^δ in H_i be multiplied by any operator which is the $p^{\alpha_{\beta-1}-\alpha'_{\beta-1}}$ power of an independent generator, then under I the operator formed by this product is conjugate with any of the operators formed by multiplying s into at least the $p^{\alpha_{\beta-1}-\alpha'_{\beta-1}}$ power of all the independent generators whose orders are less than $p^{\alpha_{\beta-1}}$ ($\beta = i+1, i+2, \cdots, n$).

The number of characteristic subgroups in G is given in the AMERICAN JOURNAL OF MATHEMATICS, Vol. 27, p. 23, and as we see does not depend on the number of independent generators of each order, but only on the different orders of the independent generators, while the orders of these characteristic subgroups are dependent on the number of independent generators of each order, as well as the orders of the independent generators.

3. SUCCESSIVE SUBGROUPS OF G INVARIANT UNDER I.

We shall represent a series of subgroups of G, whose orders are $1, p, p^2, p^3, \cdots, p^m$ respectively, such that each subgroup except the last is included in the one which follows it, and each is invariant under a given Sylow subgroup, whose order is a power of p in the I of G, by

$$G_0, G_1, G_2, G_3, \cdots, G_m \quad (G_m \equiv G). \tag{2}$$

Let S represent this Sylow subgroup, and suppose the order of S to be p^M. Such a Sylow subgroup is sometimes called the group of p-isomorphisms of G.*

Furthermore we shall suppose this series of subgroups to be selected by the method commonly used, viz., G_1 is any one of the subgroups of order p generated by an independent generator of highest order, and $G_0, G_1, G_2, \cdots, G_{\lambda_1}$ contain all the operators of order p generated respectively by the

* Cf. the first footnote.

$1, 2, 3, \cdots, \lambda_1$ independent generators of highest order, and G_{λ_1+1} is generated by G_{λ_1} and the subgroup of order p generated by an arbitrary independent generator of next largest order. Continue this arrangement so that all the operators of order p appear first, followed by all the operators of orders $p^2, p^3, p^4, \cdots, p^{\alpha_1}$ in the order named.

Such a series of subgroups is called a principal series of subgroups corresponding to the Sylow subgroup S, and it is known that S has only one principal series of subgroups.

It is evident that the successive characteristic subgroups in the principal series $G_0, G_1, G_2, \cdots, G_m$ are the subgroups composed of the identity and all the operators of order p in $\{H_1\}, \{H_1, H_2\}, \cdots, \{H_1, H_2, H_3, \cdots, H_n\}$ respectively, and in general all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_i\}$ multiplied by all the operators whose orders are less than p^δ in $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$.

Thus the following subgroups of the principal series (2) are characteristic subgroups

$$(3) \qquad \begin{gathered} G_{\lambda_1},\ G_{\lambda_1+\lambda_2},\ G_{\lambda_1+\lambda_2+\lambda_3},\ \cdots,\ G_{\lambda_1+\lambda_2+\cdots+\lambda_n},\ \cdots, \\ G_{\delta\lambda_1+\delta\lambda_2+\cdots+\delta\lambda_i+\alpha_{i+1}\lambda_{i+1}+\cdots+\alpha_n\lambda_n},\ \cdots \quad (\alpha_i \geqq \delta > \alpha_{i+1}), \end{gathered}$$

and no other subgroup of this principal series is characteristic.

It should be noted that these subgroups are not all the characteristic subgroups of G, and that in general not all the operators of one of the characteristic subgroups in (3), not in the preceding characteristic subgroup, are conjugate under the group of isomorphisms of G.

We shall consider some of the conditions in order that a series of subgroups of G whose orders are $1, p, p^2, \cdots, p^m$ respectively, and each subgroup except the last is included in the one which follows it, shall have each subgroup invariant under the I of G.

Since all the operators of the same order in H_i $(i = 1, 2, 3, \cdots, n)$ are conjugate under I, it follows that *such a series can exist only when there is but one independent generator of each order.* We shall show this condition to be sufficient. The cyclic group of order $p^{\alpha_1-\alpha_2}$ generated by the independent generator of largest order has each subgroup invariant under I. Extend this subgroup by an operator of order p generated by the independent generator of next largest order. Extend this latter subgroup by an operator which is the p^{α_2-1} power of the independent generator of largest order, and continue by an operator which is the p^{α_2-2} power of the independent generator of next largest order. If $\alpha_2 - \alpha_3 > 1$ we continue with an operator which is the p^{α_2-2} power of the independent generator of largest order. Continue this process until all the operators whose orders divide $p^{\alpha_2-\alpha_3}$ of H_2 have been used. Then continue with an operator of order p in H_3 and repeat

as above. It is evident that this process may be continued until we have all the operators of G, and that each successive subgroup satisfies the necessary and sufficient condition stated above that a subgroup be characteristic.

Furthermore in any such series each subgroup of the series must satisfy the necessary and sufficient condition, stated above, that a subgroup of G be characteristic. That the series may be chosen in many different ways is shown by the following example. Let G be of type $(m-1, 1)$, then the series of subgroups of orders $1, p, p^2, \cdots, p^m$ respectively, such that each is invariant under the I of G, may be selected in $m-2$ ways, for we may extend the cyclic subgroup of order p^δ by the independent generator of order p for $\delta = 1, 2, 3, \cdots, m-2$.

If G contains but one independent generator of largest order, then G may contain characteristic subgroups of every order less than p^m, even though some of the independent generators are of the same order. But these characteristic subgroups are such that they do not form a series of subgroups each of which is included in the one which follows. That there be but one independent generator of largest order follows from the fact that the fundamental characteristic subgroup must be of order p, since all the operators of the fundamental characteristic subgroup, besides the identity, are conjugate under I.

A simple illustration would be the group of type $(m-2, 1, 1)$ $(m \geqq 4)$. If the characteristic cyclic subgroup of order p^δ be extended by one of the independent generators of order p, the resultant group will not be characteristic. That is, the characteristic cyclic subgroup of order p^δ when extended by one of the independent generators of order p must be extended by both of the independent generators of order p, if the resultant group is to be characteristic. Since for a cyclic characteristic subgroup δ may assume the following values $1, 2, 3, \cdots, m-3$, we have characteristic subgroups of the following orders $p, p^2, p^3, \cdots, p^{m-3}$. If at least the characteristic subgroup of order p^3 and of type $(1, 1, 1)$ be extended by operators of order $p^2, p^3, \cdots, p^{m-2}$ successively, then we have characteristic subgroups of orders $p^3, p^4, p^5, \cdots, p^m$. Thus we see that this group does contain characteristic subgroups of each order but these characteristic subgroups are such that they do not form a series of subgroups each of which is included in the one which follows.

A series of subgroups $G_0, G_1, G_2, \cdots, G_m$ such that each subgroup except the last is included in the one which follows and each is invariant under a given Sylow subgroup S of the I of G may however be selected in many different ways other than being selected as the principal series. Since for a given Sylow subgroup S in the I of G there exists a series of subgroups,

such that each operator of G is transformed under S into itself multiplied by an operator in the subgroup that precedes it, then the independent generators of like orders are not conjugate under S but each is transformed into itself multiplied by at least those independent generators that precede. Thus all the independent generators of G have a definite order of arrangement for a given S and this order, as well as the order of arrangement of the respective powers of the independent generators, must be preserved in the formation of a series.

In order that any subgroup of G which includes the operators $s'_1, s'_2, s'_3, \cdots, s^1_\alpha$ $(\alpha \leqq \lambda_i)$ of order p^δ $(\delta \leqq \alpha_i)$ in H_i, which are generated by α distinct independent generators of order p^{α_i}, be invariant under S, it is necessary that this subgroup at least contains all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$ as well as all the operators generated by $s'_1, s'_2, s'_3, \cdots, s^1_\alpha$, and all the remaining operators of H_i whose orders are less than p^δ, and all the operators in H_β whose orders do not exceed $p^{\alpha'_\beta}$, where $\alpha'_{\beta-1} \geqq \alpha'_\beta \geqq \alpha_\beta - \alpha_{\beta-1} + \alpha'_{\beta-1}$ $(\alpha'_\beta \leqq \alpha'_\beta)$ as β takes successively the values $i+1, i+2, i+3, \cdots, n$ $(\alpha'_i \equiv \delta)$, $(\delta > \alpha_\beta)$.

If there is but one independent generator of each order, then any series of subgroups such that each subgroup except the last is included in the one which follows and each is invariant under I, is also invariant under S and conversely. In any group the fundamental characteristic subgroup is uniquely determined for a given Sylow subgroup in the I of G. If $\alpha_1 - \alpha_2 > 1$, the fundamental characteristic subgroup may be extended by an operator of order p^2 generated by the first independent generator of largest order or by an operator of order p generated by the first independent generator of next largest order.

The number of different ways of selecting a series of subgroups such that each subgroup except the last is included in the one which follows and each is invariant under a Sylow subgroup S of the I of G, may be best illustrated by the following examples. If G is of type $(m-2, 1, 1)$, then the number of different series is $[(m-2)(m-3)]/2!$. This follows at once for if we consider the cyclic group of order p^{m-2} to have each successive subgroup of orders $p, p^2, p^3, \cdots, p^{m-2}$ generated by $s_1, s_2, s_3, \cdots, s_{m-2}$ respectively and the two independent generators of order p to be s_{m-1} and s_m, then each successive subgroup of the series may be generated respectively by these m generators. We see that s_1 is in the fundamental characteristic subgroup and hence must generate G_1, s_{m-2} must be an operator in G_m not in G_{m-1}, and the remaining $m-2$ operators may have all possible arrangement such that $s_2, s_3, \cdots, s_{m-3}$, and s_{m-1}, s_m appear in the order named. Thus in this case the number of different ways of selecting a series of subgroups each of which is invariant under S, is the number of permutations of $m-2$

things taken two at a time in a fixed order, and hence this number is $[(m-2)(m-3)]/2!$. In these $[(m-2)(m-3)]/2!$ different series it is to be noted that the series generated by $s_1, s_{m-1}, s_m, s_2, s_3, \cdots, s_{m-2}$ respectively is the principal series corresponding to S. In the simple case when G is of type $(1, 1, 1, \cdots)$ the series is uniquely determined for each Sylow subgroup S in the I of G.

4. Operators of G which are Conjugate under I.

Let us now consider the characteristic subgroup generated by all the operators of order p^δ in G. If not all the operators of order p^δ are conjugate under I, then we shall consider some of the conditions in order that a subset of the operators of order p^δ will generate a second characteristic subgroup, such that all the operators in the first subgroup which are not in the second subgroup are conjugate under I. We shall also consider the condition in order that a subset of operators of this second characteristic subgroup will generate a third characteristic subgroup, such that all the operators in the second subgroup which are not in the third are conjugate under I.

From the condition given above for a subgroup of G to be characteristic it follows that *a necessary and sufficient condition that this first conjugate set of operators of order p^δ under I be composed of the operators formed by multiplying all the operators of order p^δ in H_i into the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$, is that p^δ be the order of the independent generators in H_i $(i < n)$*, and *a necessary and sufficient condition that this second conjugate set of operators of order p^δ under I includes all the operators of order p^δ in H_{i-1}, is that the independent generators of H_{i-1} be exactly p times as large as the independent generators of H_i.*

The first follows from the fact that by hypothesis all the operators of H_β $(\beta = i+1, i+2, \cdots, n)$ are to be included in the operators of H_β whose orders do not exceed $p^{\alpha_\beta - \alpha_i + \delta}$, hence

$$\alpha_\beta - \alpha_i + \delta = \alpha_\beta, \qquad \text{and} \qquad \delta = \alpha_i.$$

The second follows from the fact that all the operators of H_i whose orders do not exceed $p^{(\alpha_i - \alpha_{i-1} + \delta)}$ must include the operators of order $p^{\alpha_{i-1}}$ hence

$$\alpha_i - \alpha_{i-1} + \delta = \delta - 1, \qquad \text{and} \qquad \alpha_{i-1} - \alpha_i = 1 \quad (\delta \equiv \alpha_i).$$

From what precedes if $\delta = \alpha_i$, then all the operators of order p^{α_i} in G, which are formed by multiplying all the operators of order p^{α_i} in H_i into the group generated by all the operators whose orders do not exceed p^{α_i} in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$ $(i = 1, 2, 3, \cdots, n)$, are conjugate under I.

We see from this that any operator in $G_{m-\lambda_1+1}$ not in $G_{m-\lambda_1}$ corresponds to $p^m - p^{m-\lambda_1}$ operators under the group of isomorphisms of G, and for each of these correspondences any operator in $G_{m-\lambda_1+2}$ not in $G_{m-\lambda_1+1}$ can correspond to $p^m - p^{m-\lambda_1+1}$ operators under I, etc. If G_{m_2} is the subgroup of G in the principal series $G_0, G_1, G_2, \cdots, G_m$ which contains all the operators of G whose orders do not exceed p^{α_2}, then any operator in $G_{m_2-\lambda_2+1}$ not in $G_{m_2-\lambda_2}$ corresponds to $p^{m_2} - p^{m_2-\lambda_2}$ operators under I, independent of any previous correspondence.

Since G_{m_2} contains all the operators of G whose orders do not exceed p^{α_2} it is evident that in this principal series G_{m_2} precedes G_m by $(\alpha_1 - \alpha_2)\lambda_1$, viz.,

$$m_2 = m - (\alpha_1 - \alpha_2)\lambda_1 = (\lambda_1 + \lambda_2)\alpha_2 + \lambda_3\alpha_3 + \cdots + \lambda_n\alpha_n.$$

In general G_{m_i} contains all the operators of G whose orders do not exceed p^{α_i} $(i = 1, 2, 3, \cdots, n)$ $(m_1 \equiv m)$, and in the principal series (2) G_{m_i} precedes $G_{m_{i-1}}$ by $(\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1})(\alpha_{i-1} - \alpha_i)$, viz.,

$$\begin{aligned} m_i &= m_{i-1} - (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{i-1})(\alpha_{i-1} - \alpha_i) \\ &= (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_i)\alpha_i + \lambda_{i+1}\alpha_{i+1} + \cdots + \lambda_n\alpha_n. \end{aligned} \tag{4}$$

With this notation the order of the group of isomorphisms of G is the total product of

$$(p^{m_i} - p^{m_i-\lambda_i})(p^{m_i} - p^{m_i-\lambda_i+1}) \cdots (p^{m_i} - p^{m_i-1}) \quad (i = 1, 2, \cdots, n). \tag{5}$$

It has been stated that the automorphisms in which the operators of G correspond to themselves multiplied by operators in the subgroup that precedes are p-isomorphisms. Hence we see that $p^{m_1-\lambda_1}$ of the $p^{m_1} - p^{m_1-\lambda_1}$ automorphisms are p-isomorphisms, and $p^{m_1-\lambda_1+1}$ of the $p^{m_1} - p^{m_1-\lambda_1+1}$ automorphisms are p-isomorphisms, etc., such that these p-isomorphisms leave each subgroup of the principal series (2) invariant, and thus they generate the Sylow subgroup S of order p^M of the group of isomorphisms of G. The value of M is

$$\begin{aligned} M = (m_1 - 1) + (m_1 - 2) + \cdots + (m_1 - \lambda_1) + (m_2 - 1) \\ + \cdots + (m_n - \lambda_n).* \end{aligned} \tag{6}$$

It is to be noted however that of the $p^{m_1} - p^{m_1-\lambda_1}$ automorphisms the $p^{m_1-\lambda_1}$ p-isomorphisms so chosen do not necessarily include all the p-isomorphisms in this set but only those which are included in a given Sylow subgroup S.

5. Major Co-sets Defined and their Application.

In the following discussion we shall frequently have occasion to refer to the operators of a subgroup of the principal series $G_0, G_1, G_2, \cdots, G_m$ which are not in the preceding subgroup.

* G. A. Miller, American Journal of Mathematics, Vol. 36, p. 49.

The notation $G_\alpha - G_{\alpha-1}$ is frequently used in referring to the operators of G_α which are not contained in $G_{\alpha-1}$ ($\alpha = 1, 2, \cdots, m$). In this article we shall use the term *major co-set* when referring to these operators and shall represent it by $G_{\alpha-1}s_\alpha$, where s_α is any operator of $G_\alpha - G_{\alpha-1}$. However in some cases it will be more convenient to represent this major co-set by C_α.

If s be any operator of order $g \cdot n$ and G be any group which contains the cyclic subgroup of order g generated by s, then with G the operators of the quotient group of the cyclic group of order $g \cdot n$ with respect to the cyclic subgroup of order g will generate $n - 1$ co-sets, such that only $\varphi(n)$ of these co-sets will contain operators of order $g \cdot n$ where $\varphi(n)$ is the number of natural numbers which do not exceed n and are prime to n. Thus in general *we may define a major co-set Gs, G being any group containing the cyclic subgroup of order g generated by s, and s is any operator of order $g \cdot n$, as the totality of operators formed by multiplying all the operators of G successively by an operator in each of the $\varphi(n)$ distinct co-sets which contains operators of order $g \cdot n$ generated by s.*

Since every Sylow subgroup of order p^M in the I of G gives rise to one and only one principal series of subgroups, it follows that the total number of such Sylow subgroups in I is the total number of principal series that may be selected from G.

If we suppose the subgroup $G_{m_1-\lambda_1}$ of order $p^{m_1-\lambda_1}$ to contain all the operators of G whose orders are less than p^{α_1}, then the quotient group of G with respect to $G_{m_1-\lambda_1}$ is of order p^{λ_1} and of type $(1, 1, 1, \cdots)$, and the major co-set $G_{m_1-\lambda_1}s_{m_1-\lambda_1+1}$ may be selected from the operators of $G - G_{m_1-\lambda_1}$ in as many distinct ways as a subgroup of order p may be selected from this quotient group, or in $(p^{\lambda_1} - 1)/(p - 1)$ distinct ways. We next form the quotient group of G with respect to $G_{m_1-\lambda_1+1}$ and it is of order p^{λ_1-1} and of type $(1, 1, 1, \cdots)$, and the major co-set $G_{m_1-\lambda_1+1}s_{m_1-\lambda_1+2}$ may be selected from the remaining $p^{m_1} - p^{m_1-\lambda_1+1}$ operators in as many distinct ways as a subgroup of order p may be selected from this second quotient group, or in $(p^{\lambda_1-1} - 1)/(p - 1)$ distinct ways. If we continue this process for each successive major co-set we see that the subgroups of orders $p^{m_1-\lambda_1+1}$, $p^{m_1-\lambda_1+2}$, $\cdots$, p^{m_1-1}, p^m may be selected respectively in

$$\frac{p^{\lambda_1} - 1}{p - 1}, \frac{p^{\lambda_1-1} - 1}{p - 1}, \cdots, \frac{p^2 - 1}{p - 1}, 1$$

distinct ways each of which is in a distinct principal series.

Since $G_{m_i} - G_{m_i-\lambda_i}$ includes all the independent generators of G of order p^{α_i} we see that any automorphism of the operators of $G_{m_i} - G_{m_i-\lambda_i}$ ($i = 1, 2, 3, \cdots, n$) will completely determine the automorphism of G, thus we need only to consider the distinct ways of selecting the subgroups $G_{m_i-\lambda_i+1}$,

$G_{m_i-\lambda_i+2}, \cdots, G_{m_i}$ in order to determine the total number of principal series in G.

The quotient group of G_{m_i} with respect to $G_{m_i-\lambda_i}$ is of order p^{λ_i} and of type $(1, 1, 1, \cdots)$, and hence the major co-set $G_{m_i-\lambda_i}s_{m_i-\lambda_i+1}$ may be selected from the operators of $G_{m_i} - G_{m_i-\lambda_i}$ in as many distinct ways as a subgroup of order p may be selected from this quotient group, or in $(p^{\lambda_i} - 1)/(p - 1)$ distinct ways. We may continue as above and thus determine the total number of distinct principal series in G. These results may be stated in the following theorem:

THEOREM I. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order p^{α_2}, $\cdots$, λ_n of order p^{α_n} $(\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0)$, then the total number of Sylow subgroups of order p^M in the group of isomorphisms of G is the total product of*

$$\frac{p^{\lambda_1} - 1}{p - 1} \cdot \frac{p^{\lambda_2} - 1}{p - 1} \cdot \cdots \cdot \frac{p^{\lambda_n} - 1}{p - 1}$$

$$\frac{p^{\lambda_1-1} - 1}{p - 1} \cdot \frac{p^{\lambda_2-1} - 1}{p - 1} \cdot \cdots \cdot \frac{p^{\lambda_n-1} - 1}{p - 1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{p - 1}{p - 1} \cdot \frac{p - 1}{p - 1} \cdot \cdots \cdot \frac{p - 1}{p - 1}.$$

From theorem I and formulæ (5) and (6) we have the following:

THEOREM II. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order p^{α_2}, $\cdots$, λ_n of order p^{α_n} $(\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0)$, then the order of the group of isomorphisms I of G is $(p - 1)^{\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_n}$ multiplied by the product of the order of a Sylow subgroup, whose order is a power of p in I, by the total number of such Sylow subgroups in I.*

From theorems I and II we have that *a necessary and sufficient condition that the group of isomorphisms of an abelian group G of order p^m shall have for its order a power of a prime number, is that $p = 2$ and that there be but one independent generator of each order in a set of independent generators of G.*

6. OPERATORS OF G INVARIANT UNDER A SYLOW SUBGROUP OF I.

If any two operators t_1 and t_2 of the group of isomorphisms of an abelian group G transform a given operator of G into itself multiplied by s_1 and s_2 respectively, then the necessary and sufficient condition that t_1 and t_2 be commutative is that the commutator of t_1 and s_2 equals the commutator of t_2 and s_1.*

If we suppose the successive subgroups of the principal series (2) to be

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 36, p. 47.

generated by 1, s_1, s_2, s_3, $\cdots$, s_m respectively, then we may suppose the set of independent generators of G in this discussion to be

$$(7) \qquad s_{m_n-\lambda_n+1},\ s_{m_n-\lambda_n+2},\ \cdots,\ s_{m_n},\ s_{m_{n-1}-\lambda_{n-1}+1},\ \cdots,\ s_{m_1}.$$

Furthermore, in the remaining discussion, by s_α we shall mean an operator of the set of independent generators given in (7). Hence s_α is transformed, under a Sylow subgroup S of I, into itself multiplied by every operator of the subgroup which precedes s_α. If this subgroup contains an operator s'_α of the same order as s_α, then there is no power of s_α, unless it reduces to the identity, such that all the operators of S shall be commutative with this power. For if t_1 is an operator of S such that

$$t_1^{-1} s_\alpha t_1 = s'_\alpha s_\alpha,$$

then

$$t_1^{-1} s_\alpha^n t_1 = s_\alpha'^n s_\alpha^n.$$

Since in a principal series of subgroups corresponding to S an operator s_α is preceded by the operators of the same order as s_α, which are generated by the independent generators of orders greater than that of s_α, providing s_α is not an independent generator of largest order, it follows that the only condition that not one of the commutators of s_α under S can be of the same order as s_α is that s_α be the first independent generator of largest order in the principal series corresponding to S.

That S cannot be commutative with the operators generated by s_α which are of order greater than p, follows from the fact that, under S, s_α may have for its commutator an operator whose order is one pth the order of s_α.

Thus in any abelian group G of order p^m, the subgroup G_1 in the principal series of subgroups G_0, G_1, G_2, $\cdots$, G_m corresponding to S contains all the operators of G which are invariant under the Sylow subgroup S.

7. Invariant Operators in a Sylow Subgroup of I.

It is known that the invariant operators of I are those which transform every operator of G into the same power of itself. Of this number only those which transform every operator of G into itself multiplied by some power of p of itself are p-isomorphisms, and these p-isomorphisms are common to every Sylow subgroup, whose order is a power of p, in I. It will be assumed that this condition is sufficient and it will be shown that this necessarily includes all the operators in any Sylow subgroup S which are invariant under S, when not all the invariants of G are equal to p. In case G is of type $(1, 1, 1, \cdots)$, then a necessary and sufficient condition that an operator t_1 in a Sylow subgroup S of order p^m in I be invariant under S is that it be commutative with every operator of G_{m-1}, and the commutators

of t_1 and the operators of the major co-set $G_{m-1}s_m$ must be the operators of order p in G_1. For if

$$t_1^{-1}s_m t_1 = s_1 s_m, \qquad t_2^{-1}s_m t_2 = s_\beta s_m,$$

where t_2 is any operator in the Sylow subgroup containing t_1, then we have

$$(t_1 t_2)^{-1}s_m t_1 t_2 = s_1 s_\beta s_m, \qquad (t_2 t_1)^{-1}s_m t_2 t_1 = s_1 s_\beta s_m.$$

It is evident that the condition is sufficient. This is equivalent to transforming every operator of G into itself multiplied by the p^{α_1-1} power of itself in case G is not of type $(1, 1, 1, \cdots)$. The following proof for the necessary condition is valid for any prime power abelian group. Suppose that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha \quad (s'_\alpha \neq s_1,\ s'_\alpha \neq s_\alpha^{kp},\ a < m),$$

$$t_3^{-1}s_\alpha t_3 = s_\alpha, \qquad t_3^{-1}s'_\alpha t_3 = s_\beta s'_\alpha \quad (s_\beta \neq 1),$$

then

$$(t_1 t_3)^{-1}s_\alpha t_1 t_3 = s_\beta s'_\alpha s_\alpha, \qquad (t_3 t_1)^{-1}s_\alpha t_3 t_1 = s'_\alpha s_\alpha.$$

Next consider the case $s'_\alpha = s_1$. In this case t_1 either transforms all of the operators of the set of independent generators of G given by (7) into themselves multiplied by operators of order p in G_1 or is commutative with some of these generators. The possibility of t_1 transforming some of these operators into themselves multiplied by some pth power of itself is completely discussed in the next case. Let

$$t_1^{-1}s_\alpha t_1 = s_1 s_\alpha, \qquad t_4^{-1}s_\alpha t_4 = s_\sigma s_\alpha,$$

then

$$(t_1 t_4)^{-1}s_\alpha t_1 t_4 = s_1 s_\sigma s_\alpha, \qquad (t_4 t_1)^{-1}s_\alpha t_4 t_1 = s_1^2 s_\sigma s_\alpha.$$

If t_1 is commutative with some of the operators of G, say s_α, but is not commutative with s_σ, then

$$(t_1 t_4)^{-1}s_\alpha t_1 t_4 = s_\sigma s_\alpha, \qquad (t_4 t_1)^{-1}s_\alpha t_4 t_1 = s_1 s_\sigma s_\alpha.$$

The next case is that s'_α be some pth power of s_α or $s'_\alpha = s_\alpha^{kp}$. If we assume that t_1 does not transform every operator of G into itself multiplied by some pth power of itself, and suppose s_σ to be this operator, then we have

$$(t_1 t_4)^{-1}s_\alpha t_1 t_4 = s_\sigma^{kp} s_\alpha^{kp} s_\sigma s_\alpha, \qquad (t_4 t_1)^{-1}s_\alpha t_4 t_1 = t_1^{-1}s_\sigma t_1 s_\alpha^{kp} s_\alpha,$$

and thus establishing the necessary condition.

It follows from the known formula

$$t^{-n}s_\alpha t^n = s_{\alpha-n}\, s_{\alpha-n+1}^{n} \cdots s_{\alpha-n+r}^{[n(n-1)\cdots(n-r+1)]/r!} \cdots s_{\alpha-1}^{n} s_\alpha, \tag{8}$$

where $t^{-1}s_\beta t = s_{\beta-1}s_\beta$ $(\beta = \alpha,\ \alpha-1,\ \alpha-2,\ \cdots,\ \alpha-n+1)$, that if $t_1^{-1}s_m t_1 = s_m^p s_m$, then t_1^p would transform s_m into itself multiplied by an operator in the major co-set containing $s_m^{p^2}$, and hence $t_1^{p^{\alpha_1-1}} \equiv 1$. In case G

is of type (1, 1, 1, $\cdots$),

$$t_1^{-1}s_m t_1 = s_1 s_m, \qquad \text{and} \qquad t_1^{-p}s_m t_1^p = s_1^p s_m,$$

and hence t_1 is of order p.

From what precedes we see that *the invariant operators of a Sylow subgroup S of order p^m in the I of G generate a cyclic subgroup of order p^{α_1-1} for $\alpha_1 > 1$, and generate a subgroup of order p for $\alpha_1 = 1$. The first being composed of the operators which transform every operator of G into itself multiplied by some pth power of itself, and the second is composed of the operators which are commutative with G_{m-1} and transform the operators of the major co-set $G_{m-1}s_m$ into themselves multiplied by operators of order p in G_1, and hence a Sylow subgroup whose order is a power of p in the group of isomorphisms of any abelian group G of order p^m contains but one invariant subgroup of order p.*

8. Subgroups Common to the Sylow Subgroups of I.

Since in the principal series of subgroups corresponding to any Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, the subgroup $G_{m_i-\lambda_i}$ of order $p^{m_i-\lambda_i}$ will always precede all the operators of $G_{m_i} - G_{m_i-\lambda_i}$, it is evident that the automorphisms of G which make the operators of $G_{m_i} - G_{m_i-\lambda_i}$ correspond to themselves multiplied by operators in $G_{m_i-\lambda_i}$ are common to every such Sylow subgroup in I, and that no other automorphisms of these operators are common to every such Sylow subgroup ($i = 1, 2, 3, \cdots, n$). We see this from the fact that G_{m_i} is a characteristic subgroup, and hence any p-isomorphism which makes an operator s_σ of the major co-set $G_{\sigma-1}s_\sigma$ corresponds to itself multiplied by an operator s_β in the major co-set $G_{\beta-1}s_\beta$ ($m_i - \lambda_i < \beta < \sigma \leqq m_i$) is not in the Sylow subgroup to which the principal series generated by 1, s_1, s_2, s_3, $\cdots$, $s_{m_i-\lambda_i}$, $\cdots$, s_σ, s_β, $\cdots$, s_m corresponds.

Thus we have *the order of the subgroup common to every Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, to be $p^{M'}$,*

$$M' = \sum_1^n (m_i - \lambda_i)\lambda_i.$$

From (6) we see that *the index of this subgroup with respect to S is $p^{M-M'}$,*

$$M - M' = \frac{1}{2}\sum_1^n (\lambda_i - 1)\lambda_i.$$

9. Invariant Subgroups in a Sylow Subgroup of I.

Consider the complete set of conjugates of an operator t_1 of S under the operators of S. Let t be any other operator of S, and suppose

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1}s_\alpha t = s_\beta s_\alpha, \qquad t_1^{-1}s_\beta t_1 = s'_\beta s_\beta,$$

then

$$(tt_1t^{-1})^{-1}s_\alpha tt_1t^{-1} = tt_1^{-1}t^{-1}s_\alpha tt_1t^{-1} = ts'_\beta s'_\alpha s_\beta s_\alpha t^{-1} = ts'_\beta s'_\alpha t^{-1}s_\alpha.$$

If t be allowed to vary through all the operators of S, we see that s_β varies through all the operators of $G_{\alpha-1}$ and s'_β varies through all the commutators of $G_{\alpha-1}$ and t_1. From the expression $ts'_\beta s'_\alpha t^{-1}$ it follows that the complete set of conjugates of t_1 under S transforms s_α into itself multiplied by every operator of the major co-set containing s'_α whenever s'_α is not in the commutator subgroup of t_1 and $G_{\alpha-1}$, and in all other cases by every operator in the commutator subgroup of t_1 and $G_{\alpha-1}$.

Hence *a necessary condition that a subgroup in a Sylow subgroup S in I be invariant is that the commutators of the operators of G under the operators of this subgroup generate a group which is one of the subgroups of the principal series corresponding to S.* This is necessary for otherwise this subgroup would not contain all of its conjugates under S. However this condition is not sufficient as may be illustrated by the following example.

Suppose G to be of order p^4 and of type (1, 1, 1, 1), and suppose the principal series G_0, G_1, G_2, G_3, G_4, corresponding to S, to be generated by 1, s_1, s_2, s_3, s_4, respectively, then the two automorphisms t_1 and t_2 of G, which are established by making

$$t_1^{-1}s_1t_1 = s_1, \quad t_1^{-1}s_2t_1 = s_2, \quad t_1^{-1}s_3t_1 = s_2s_3, \quad t_1^{-1}s_4t_1 = s_4,$$
$$t_2^{-1}s_1t_2 = s_1, \quad t_2^{-1}s_2t_2 = s_2, \quad t_2^{-1}s_3t_2 = s_2s_3, \quad t_2^{-1}s_4t_2 = s_1s_4,$$

together with the invariant subgroup of order p, will generate an abelian subgroup of order p^3. Yet this subgroup is not invariant under S even though the commutator subgroup of G under this subgroup is G_2. This subgroup is abelian because t_1 and t_2 are commutative with G_2. That this subgroup is not invariant under S is at once evident if we transform it by an operator t of S, which is commutative with s_1, s_3, and s_4 and $ts_2t^{-1} = s_1s_2$, hence

$$(tt_1t^{-1})^{-1}s_3tt_1t^{-1} = ts_2s_3t^{-1} = s_1s_2s_3$$

which is not an operator of this subgroup.

It should be noted however that the subgroup of S, composed of all the operators of S which transform the operators of G into themselves multiplied by the operators of any one of the principal series of subgroups corresponding to S, is invariant under S, but not all the operators of this subgroup are conjugate.

If a Sylow subgroup S of I be generated by operators that are commutative with every independent generator of G given by (7) *save s_j, and transform s_j into itself multiplied by s_i where j takes successively the values m_1, $m_1 - 1$, $\cdots$, $i + 1$ for each s_i and i takes successively the values* 1, 2, 3, $\cdots$, $m - 1$, *then each sub-*

group of order p, p^2, p^3, p^4, $\cdots$, p^M respectively is an invariant subgroup of S. If s_{i+1} is not an independent generator of G, then s_j varies through only the independent generators given by (7) *which are not in G_{i+1}.*

It is evident that any one of these subgroups is composed of all the operators of S which transform the independent generators $s_{m_1}, s_{m_1-1}, \cdots, s_j$ into themselves multiplied by the operators of G_i, which transform the independent generators $s_{j-1}, \cdots, s_i$ into themselves multiplied by the operators of G_{i-1}, and which transform each of the remaining independent generators of G into itself multiplied by any operator that precedes it. If t_1 is one of these operators and t any other operator of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1}s_\alpha t = s_\beta s_\alpha, \qquad t_1^{-1}s_\beta t_1 = s'_\beta s_\beta,$$

then

$$(tt_1t^{-1})^{-1}s_\alpha tt_1t^{-1} = ts'_\alpha s'_\beta t^{-1}s_\alpha.$$

Since s_β is in $G_{\alpha-1}$ we see that all the conjugates of t_1 under S transform s_α into itself multiplied by operators of the co-set containing s'_α and hence is in this subgroup.

If the Sylow subgroup S of I be generated by operators which are commutative with every independent generator of G save s_j, and transform s_j into itself multiplied by s_i where i takes successively the values 1, 2, 3, $\cdots$, $j-1$ *for each s_j, and j takes successively the values m_1, m_1-1, $\cdots$, $m_1-\lambda_1+1$, m_2, $\cdots$, $m_n-\lambda_n+1$, then each subgroup of order p, p^2, p^3, $\cdots$, p^M respectively is an invariant subgroup of S.*

From the manner in which this Sylow subgroup is generated we see that there exists a subgroup containing t_1 which transforms any independent generator s_α into itself multiplied by all the operators of the commutator subgroup of t_1 and G_α independent of any automorphism of the remaining independent generators, and hence all the conjugates of t_1 under S are in this subgroup and thus it is an invariant subgroup.

The subgroup of order p in the two cases is common. The invariant subgroups of S of orders p^2, p^3, $\cdots$, p^{m-1} respectively are distinct in the two cases, because in the first case the operators in any one of these invariant subgroups are not commutative with the independent generator s_{m-1} of G, and in the second case all the operators of any one of these invariant subgroups are commutative with every independent generator of G except s_m. In the remaining invariant subgroups of orders p^m, p^{m+1}, $\cdots$, p^{M-1} respectively the operators in the first case do not give rise to commutators in the co-set containing s_{m-1}, while the operators in the second case for each of these invariant subgroups give rise to commutators in the co-set containing s_{m-1}. *Hence a Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, contains at least $p+1$ invariant subgroups of each order greater than p.*

Furthermore *a subgroup of S composed of all the operators of S which transform each operator of the principal series corresponding to S into itself multiplied by an operator in at least the* βth *major co-set which precedes the major co-set containing this operator, being commutative with the operators of G_β, is invariant under S.* For if t be any operator in this subgroup and t_1 any other operator of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1}s_\alpha t = s_{\alpha-\beta}s_\alpha, \qquad t^{-1}s'_\alpha t = s'_{\alpha-\beta}s'_\alpha,$$

then

$$(t_1 t t_1^{-1})^{-1}s_\alpha t_1 t t_1^{-1} = t_1 s'_{\alpha-\beta}s_{\alpha-\beta}t_1^{-1}s_\alpha,$$

and hence all the conjugates of t under S transform s_α into itself multiplied by operators in at least the βth major co-set which precedes the major co-set containing s_α. Thus it follows that all the conjugates under S of any operator of the subgroup composed of all the operators of S which transform every operator of G into itself multiplied by an operator in at least the βth major co-set which precedes are in this subgroup, and hence this subgroup is invariant under S.

Let t_1 and t_2 be any two operators of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t_2^{-1}s_\alpha t_2 = s''_\alpha s_\alpha, \qquad t_1 s''_\alpha t_1^{-1} = s''_{\alpha-1}s''_\alpha, \qquad t_2^{-1}s'_\alpha t_2 = s'_{\alpha-1}s'_\alpha,$$

then

$$(t_1 t_2 t_1^{-1}t_2^{-1})^{-1}s_\alpha t_1 t_2 t_1^{-1}t_2^{-1} = t_2 t_1 s'_{\alpha-1}s''_\alpha s'_\alpha s_\alpha t_1^{-1}t_2^{-1} = t_2 t_1 s'_{\alpha-1}t_1^{-1}s''_{\alpha-1}t_2^{-1}s_\alpha.$$

It follows from this equation that the commutator of any two operators of S transforms any operator of G into itself multiplied by an operator in at least the second major co-set which precedes it. Thus we have that the commutator subgroup of S is included in the invariant subgroup composed of all the operators which transform the operators of G into themselves multiplied by operators in at least the second major co-set which precedes them. The order of this invariant subgroup is $p^{M-m'}$,

$$M - m' = (m_1 - 2) + (m_1 - 3) + \cdots + (m_1 - \lambda_1 - 1) + (m_2 - 2) + \cdots + (m_n - \lambda_n - 1),$$

which by the aid of (6) reduces to

$$M - m' = M - (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n), \tag{9}$$

where $m_1 - \lambda_1 - 1 = 0$ in case G is of type $(1, 1, 1, \cdots)$ and $m' = m - 1$.

If t_1 and t_2 be allowed to vary through the operators of S, we see that s'_α and s''_α will vary through all the operators of $G_{\alpha-1}$, and hence $s'_{\alpha-1}$ and $s''_{\alpha-1}$ will vary through all the possible commutators of $G_{\alpha-1}$ under S. If the major co-set $G_{\alpha-2}s_{\alpha-1}$ includes an independent generator, then the commutators of $G_{\alpha-1}$ under S will include all the operators of $G_{\alpha-2}$ and $s'_{\alpha-1}$ and $s''_{\alpha-1}$ will vary through $p^{\alpha-2}$ different operators. In case the major co-set $G_{\alpha-2}s_{\alpha-1}$ does not include an independent generator of G, then neither $s'_{\alpha-1}$ nor $s''_{\alpha-1}$ can vary through $p^{\alpha-2}$ operators.

Suppose s_α to be the first independent generator of order p^{α_i}, then the major co-set $G_{\alpha-2}s_{\alpha-1}$ will not contain an independent generator for $\alpha_i \neq \alpha_1$. An operator in this major co-set is conjugate under S with only the operators obtained by multiplying this operator by the group generated by all the operators whose orders do not exceed p^{α_i} in $H_1, H_2, H_3, \cdots, H_{i-2}$, together with all the operators whose orders do not exceed p^{α_i} in H_{i-1}, except those operators of H_{i-1} which are in the major co-set $G_{\alpha-2}s_{\alpha-1}$, and together with all the operators of H_β whose orders do not exceed $p^{\alpha_\beta - \alpha_{i-1}+\alpha_i}$ for $\beta = i, i+1, i+2, \cdots, n$. We see that the independent generators of orders less than p^{α_i} cannot be commutators under S for the operators in the major co-set $G_{\alpha-2}s_{\alpha-1}$ ($\alpha_i \neq \alpha_1$). Furthermore the pth power of the independent generators of order p^{α_i} cannot be commutators under S for the operators in the major co-set $G_{\alpha-2}s_{\alpha-1}$ if $\alpha_{i-1} - \alpha_i > 1$. If s'_α or s''_α is an independent generator of order less than p^{α_i}, then $s'_{\alpha-1}$ or $s''_{\alpha-1}$ can vary through every operator in the subgroup preceding s'_α or s''_α. In this case the pth power of the independent generators of order p^{α_i} cannot be commutators under S unless $\alpha_i - \alpha_{i+1} = 1$.

The order of the commutator subgroup of S is p^N, where

$$N = (m_1 - 2) + (m_1 - 3) + \cdots + (m_1 - \lambda_1) + (m_1 - \lambda_1 - 2) + (m_2 - 2) + \cdots + (m_2 - \lambda_2) + (m_2 - \lambda_2 - 3) + \cdots + (m_n - \lambda_n - 2),$$

where

$$\alpha_i - \alpha_{i+1} > 1 \qquad \text{for} \qquad i = 1, 2, 3, \cdots, n \quad (\alpha_{n+1} = 0).$$

If $\alpha_i - \alpha_{i+1} = 1$ for k different values of i, then k must be added to this value of N, whence from (6) N may be expressed as follows:

$$N = M - [\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n + 2(n-1) - k]. \tag{10}$$

The pth power of any operator t_1 of S is in the commutator subgroup. This follows directly from (8), for if

$$t_1^{-1} s_\alpha t_1 = s'_\alpha s_\alpha,$$

then

$$t_1^{-p} s_\alpha t_1^p = s s'^p_\alpha s_\alpha,$$

and the commutator $s s'^p_\alpha$ is evidently in the commutator subgroup, the operator s being the product of all the multiplying operators which precedes s'_α. Thus the quotient group of S with respect to the commutator subgroup is of type $(1, 1, 1, \cdots)$.

10. Total Number of p-Isomorphisms in the Group of Isomorphisms.

Let us first consider the total number of p-isomorphisms in the group of isomorphisms of the abelian group G of order p^m and of type $(1, 1, 1, \cdots)$.

We shall first consider the p-isomorphisms in I which are commutative with every operator of a given subgroup G_{m-1} of order p^{m-1}. Under these automorphisms of G an operator in the major co-set $G_{m-1}s_m$ may be made to correspond to itself multiplied by any operator in G_{m-1}, except the identity. This number is evidently

$$p^{m-1} - 1,$$

and the total number of p-isomorphisms which are commutative with p^{m-1} operators is the product of the number of p-isomorphisms commutative with a given subgroup of order p^{m-1} and the total number of subgroups of order p^{m-1} in G. Thus the total number of p-isomorphisms in I which are commutative with p^{m-1} operators is given by the following expression:

$$(11) \qquad (p^{m-1} - 1)\frac{(p^m - 1)}{(p - 1)}.$$

We now consider the p-isomorphisms commutative with a given subgroup G_{m-2} of order p^{m-2}, generated by the independent generators s_1, s_2, s_3, $\cdots$, s_{m-2}, and commutative with no other operators.

These p-isomorphisms may be divided into two classes. In the first class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-2}s_{m-1}$ give rise to commutators with which the p-isomorphisms are commutative. In the second class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-2}s_{m-1}$ give rise to commutators with which the p-isomorphisms are not commutative.

The quotient group H_2 of G with respect to G_{m-2} is of order p^2, and we may suppose it to be generated by s_{m-1}, and s_m and G to be the direct product of G_{m-2} and H_2. We see that an automorphism of any operator in a major co-set, with respect to the subgroup composed of all the commutative operators, completely establishes the automorphism of every operator of this major co-set, and hence any automorphism of the operators of H_2 will completely determine the automorphism of the group G.

For the p-isomorphisms in the first class an operator of the major co-set $G_{m-2}s_{m-1}$ can be made to correspond to itself multiplied by any operator of G_{m-2}, except the identity, or can be made to correspond to $(p^{m-2} - 1)$ operators, and for each of these automorphisms an operator in the major co-set $G_{m-1}s_m$ can be made to correspond to itself multiplied by any operator of G_{m-1}, except the operators of the subgroup of order p generated by the commutators under these p-isomorphisms of the operators of the major co-set $G_{m-2}s_{m-1}$, viz., it can have $p^{m-1} - p$ distinct automorphisms. Thus the total number of p-isomorphisms in this first class is

$$(p^{m-2} - 1)(p^{m-1} - p).$$

If the commutators of the operators of the major co-set $G_{m-2}s_{m-1}$ are not commutative with these p-isomorphisms, then they must be operators of the major co-set $G_{m-1}s_m$, and for each automorphism of an operator in the major co-set $G_{m-2}s_{m-1}$, which gives rise to one of the $p^{m-1} - p^{m-2}$ commutators in the major co-set $G_{m-2}s_m$, an operator in the major co-set $G_{m-2}s_m$ can be made to correspond to itself multiplied by any one of the $p^{m-2} - 1$ operators of G_{m-2}. Since in the quotient group H_2 we have $p + 1$ subgroups of order p, and p of these subgroups do not contain the operator s_{m-1}, we see that there are p distinct major co-sets with respect to G_{m-2} exclusive of $G_{m-2}s_{m-1}$. Hence the major co-set $G_{m-2}s_m$ may be selected from the operators of the major co-set $G_{m-1}s_m$ in p distinct ways, and the total number of p-isomorphisms in this second class is

$$(p^{m-2} - 1)(p^{m-1} - p^{m-2})p.$$

The total number of p-isomorphisms commutative with a given subgroup G_{m-2} of order p^{m-2} and commutative with no other operators of G is

$$(p^{m-2} - 1)(p^{m-1} - p) + (p^{m-2} - 1)(p^{m-1} - p^{m-2})p = (p^{m-1} - 1)(p^{m-1} - p).$$

Hence the total number of p-isomorphisms which are commutative with exactly p^{m-2} operators is

$$(12) \qquad (p^{m-1} - 1)(p^{m-1} - p)\frac{(p^m - 1)(p^{m-1} - 1)}{(p^2 - 1)(p - 1)},$$

since $\dfrac{(p^m - 1)(p^{m-1} - 1)}{(p^2 - 1)(p - 1)}$ is the number of distinct subgroups of order p^{m-2} in G, remembering that the number of subgroups of index p^α is the same as the number of subgroups of order p^α.

The number of p-isomorphisms in a given Sylow subgroup, whose order is a power of p in the group of isomorphisms of an abelian group of order p^m and of type $(1, 1, 1, \cdots)$, commutative with a given subgroup of order p^β $(\beta < m)$ and commutative with no other operators of G is given by the following expression:

$$(13) \qquad (p^\beta - 1)(p^{\beta+1} - p)(p^{\beta+2} - p^2) \cdots (p^{m-1} - p^{m-\beta-1}).$$

This is evident from the fact that any operator in the major co-set $G_{\alpha-1}s_\alpha$ $(\beta < \alpha \leqq m)$ may have for its commutator any operator in $G_{\alpha-1}$ which is not in the subgroup of order $p^{\alpha-\beta-1}$ generated by the commutators, under these p-isomorphisms, of the operators in $G_{\alpha-1} - G_\beta$. That is, for any p-isomorphisms of the operators in $G_{\alpha-1} - G_\beta$, an operator in the major co-set $G_{\alpha-1}s_\alpha$ can be made to correspond to itself multiplied by any one of the $p^{\alpha-1} - p^{\alpha-\beta-1}$ operators in $G_{\alpha-1}$ which have not been used as commutators $(\alpha = \beta + 1, \beta + 2, \cdots, m)$.

We shall now consider the p-isomorphisms commutative with a given subgroup G_{m-3}, generated by the independent generators $s_1, s_2, s_3, \cdots, s_{m-3}$, and commutative with no other operators of G. These p-isomorphisms may be divided into three classes. In the first class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators with which the p-isomorphisms are commutative. This number can be shown to be

$$\text{(i)} \quad (p^{m-3}-1)(p^{m-2}-p)(p^{m-1}-p^2) + (p^{m-3}-1)(p^{m-2}-p)(p^{m-1}-p^{m-2})p.$$

In the second class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators which themselves give rise to commutators with which the p-isomorphisms are commutative, and this number is

$$\text{(ii)} \quad (p^{m-3}-1)(p^{m-2}-p^{m-3})(p^{m-1}-p^2)\frac{(p^3-p)}{(p-1)}.$$

In the third class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators which themselves give rise to commutators with which the p-isomorphisms are not commutative, this number is

$$\text{(iii)} \quad (p^{m-3}-1)(p^{m-2}-p^{m-3})(p^{m-1}-p^{m-2})\frac{(p^3-p)(p^3-p^2)(p^2-1)}{(p^2-1)(p^2-p)(p-1)}.$$

It is evident that every p-isomorphism which is commutative with G_{m-3} and commutative with no other operator of G falls in one of these three classes. Hence the total number of p-isomorphisms in the group of isomorphisms of G which are commutative with a given subgroup G_{m-3} is the sum of (i), (ii), and (iii), which when reduced gives

$$(p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2).$$

Since the total number of subgroups of order p^{m-3} in G is

$$\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)}{(p-1)(p^2-1)(p^3-1)},$$

we have that the total number of p-isomorphisms in the group of isomorphisms of G which are commutative with exactly p^{m-3} operators is given by the following:

$$(14) \quad (p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2)\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)}{(p-1)(p^2-1)(p^3-1)}.$$

We have the total number of p-isomorphisms in I commutative with exactly p^{m-1} operators, exactly p^{m-2} operators, and exactly p^{m-3} operators.

The preceding method for determining the total number of p-isomorphisms in I becomes very complicated if we continue, as above, step by step. We shall show by mathematical induction that the law of (11), (12), and (14) holds for $m+1$ if it holds for m.

We shall assume that the number of p-isomorphisms in I commutative with exactly p^{m-r} operators is equal to

$$(15)\qquad (p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})\frac{(p^m-1)(p^{m-1}-1)\cdots(p^{m-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)},$$

and we shall show that in case G is of order p^{m+1} the number of p-isomorphisms commutative with exactly p^{m-r} operators is equal to

$$(16)\qquad (p^m-1)(p^m-p)\cdots(p^m-p^r)\frac{(p^{m+1}-1)(p^m-1)\cdots(p^{m-r+1}-1)}{(p^{r+1}-1)(p^r-1)\cdots(p-1)}.$$

Formula (15) gives the total number of p-isomorphisms commutative with a subgroup of index p^r while (16) gives the total number of p-isomorphisms commutative with a subgroup of index p^{r+1}. It is evident that (16) is obtained from (15) by replacing m by $m+1$ when r is such as to give the total number of p-isomorphisms commutative with a subgroup of index p^{r+1}, viz., for r equal to $r+1$.

From (11) we have the total number of p-isomorphisms commutative with a subgroup of index p for any value of m. Likewise (12) gives the total number of p-isomorphisms commutative with a subgroup of index p^2 for any value of m. However we shall establish the latter for the case when the group is of order p^{m+1} from (15), assuming r to be equal to one.

We shall suppose the group of order p^{m+1} to be G_{m+1} and suppose it to be generated by G and s_{m+1}. We have the number of p-isomorphisms, commutative with a given subgroup G_{m-1} in the group G, to be $(p^{m-1}-1)$. For each p-isomorphism of this group G, an operator in the major co-set Gs_{m+1} can be made to correspond to itself multiplied by any operator in G which is not in the subgroup of order p generated by the commutators of G under these automorphisms. Thus the $(p^{m-1}-1)(p^m-p)$ p-isomorphisms include all the p-isomorphisms in the group of isomorphisms of the group G_{m+1} which are commutative with a given subgroup G_{m-1} of order p^{m-1} and such that the operators in the major co-set Gs_{m+1} are not used as commutators. For the remaining p-isomorphisms, which are such that the operators of the major co-set $G_{m-1}s_m$ have for their commutators operators of the major co-set Gs_{m+1}, each set of p^m-p^{m-1} commutators may be selected from the operators of the major co-set Gs_{m+1} in $\frac{(p^2-p)}{(p-1)}$ distinct

ways. This follows from the fact that the number of distinct sets of commutators the operators of the major co-set $G_{m-1}s_m$ can have is the number of subgroups of order p in the quotient group of G_{m+1} with respect to G_{m-1} diminished by the subgroup of order p corresponding to G. An operator in each set of commutators selected from Gs_{m+1} can be made to correspond to $p^{m-1} - 1$ operators, thus the total number of these p-isomorphisms is

$$(p^{m-1} - 1)(p^m - p^{m-1})\frac{p^2 - p}{p - 1}.$$

Thus the total number of p-isomorphisms in the group of isomorphisms of G_{m+1} commutative with a given subgroup of order p^{m-1} and commutative with no other operators is

$$(p^{m-1} - 1)\left[p^m - p + (p^m - p^{m-1})\frac{p^2 - p}{p - 1}\right] = (p^m - 1)(p^m - p),$$

as was to be proved.

We shall now consider the number of p-isomorphisms commutative with a given subgroup G_{m-r} of index p^{r+1} in the group G_{m+1} and commutative with no other operators of G_{m+1} ($r = 1, 2, 3, \cdots, m - 1$). From (15) we have the number of p-isomorphisms, commutative with a given subgroup G_{m-r} of order p^{m-r} in a group G of order p^m, to be

$$(p^{m-1} - 1)(p^{m-1} - p)(p^{m-1} - p^2) \cdots (p^{m-1} - p^{r-1}).$$

For each of these p-isomorphisms of the group G, an operator in the major co-set Gs_{m+1} can have for its commutator any operator in G which is not in the subgroup of order p^r generated by the commutators of G under these p-isomorphisms. These automorphisms evidently include all the p-isomorphisms of the group of isomorphisms of the group G_{m+1} which are commutative with a given subgroup G_{m-r} and such that the operators in the major co-set Gs_{m+1} are not used as commutators.

In all the remaining p-isomorphisms of G_{m+1} commutative with G_{m-r}, it is necessary that some operators of the major co-set Gs_{m+1} be used as commutators. Since any two subgroups of index p have a subgroup of index p^2 in common, it follows that every subgroup of order p^m containing G_{m-r}, except G, contains exactly $p^m - p^{m-1}$ operators of the major co-set Gs_{m+1}. The number of distinct subgroups of order p^m, different from G, that can be selected from this group of order p^{m+1}, such that each of these subgroups of order p^m contains the given subgroup G_{m-r}, is the number of subgroups of order p^r in the quotient group of G_{m+1} with respect to G_{m-r} diminished by the subgroup of order p^r corresponding to G. The quotient group is of order p^{r+1}, and the number of subgroups of order p^r is $\frac{p^{r+1} - 1}{p - 1}$.

Therefore the number of subgroups of order p^m, except G, in G_{m+1} which contains G_{m-r} is $\dfrac{p^{r+1}-p}{p-1}$.

Let us designate the operators of $G_{m-r+1} - G_{m-r}$ by C_{m-r+1}, the operators of $G_{m-r+2} - G_{m-r+1}$ by C_{m-r+2}, $\cdots$, the operators of $G_m - G_{m-1}$ by C_m.

If in any of the $(p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})$ possible p-isomorphisms, commutative with G_{m-r}, of any of the subgroups of order p^m except G, we have the commutators of the operators of C_m to be in C_{m-1}, those of C_{m-1} to be in C_{m-2}, $\cdots$, those of $C_{m-\alpha+1}$ to be in $C_{m-\alpha}$ $(\alpha < r)$, and the subgroup $G_{m-\alpha}$ to be invariant, and that C_m, C_{m-1}, C_{m-2}, $\cdots$, $C_{m-\alpha}$ include all the operators of the major co-set Gs_{m+1} which are in the subgroup of order p^m and that each major co-set C_m, C_{m-1}, $\cdots$, $C_{m-\alpha}$ contains some operators from the major co-set Gs_{m+1}, then for each such p-isomorphism of the group of order p^m we can have an operator not in this subgroup of order p^m to have for its commutator any operator of C_m. Under some of these p-isomorphisms of the group of order p^m, the operators of this subgroup do not have for their commutators any operator of the major co-set Gs_{m+1}. In such cases C_m is such as to include all the operators of the major co-set Gs_{m+1} which are in this subgroup of order p^m, and hence $\alpha = 0$. That is, if for each p-isomorphism of any subgroup of order p^m, except G, which is commutative with a given subgroup G_{m-r} of order p^{m-r} and commutative with no other operators, we allow an operator not in this subgroup of order p^m to have for its commutator any one of the $p^m - p^{m-1}$ operators in C_m, as defined above, then it can be shown that this gives all the p-isomorphisms of the group G_{m+1} of order p^{m+1} which are commutative with G_{m-r} and such that $p^m - p^{m-1}$ operators of the major co-set Gs_{m+1} are used as commutators. Therefore the total number of p-isomorphisms in the group of isomorphisms of the group G_{m+1} of order p^{m+1} and of type $(1, 1, 1, \cdots)$ which are commutative with a given subgroup of order p^{m-r} and commutative with no other operators is given by the following:

$$(p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})\left[p^m - p^r + (p^m - p^{m-1})\frac{p^{r+1}-p}{p-1}\right],$$

which when reduced gives

$$(p^m-1)(p^m-p)\cdots(p^m-p^r). \tag{17}$$

Suppose that a p-isomorphism t of the group of isomorphisms of the group G_{m+1} which is commutative with G_{m-r} and commutative with no other operators is such that

$$t^{-1}s_\beta t = s_{\beta-1}s_\beta \qquad \text{for} \qquad \beta = \beta,\ \beta-1,\ \beta-2,\ \cdots,\ \text{and}\ \beta-\alpha,$$

and leaves $G_{m-\alpha}$ invariant. There is no loss in generality in assuming the invariant subgroup $G_{m-\alpha}$ of order $p^{m-\alpha}$ to be first. If we extend this subgroup $G_{m-\alpha}$ by $s_{\beta-\alpha}, s_{\beta-\alpha+1}, \cdots, s_\beta$ respectively, then it is evident that this p-isomorphism t, which is any p-isomorphism of the group G_{m+1} which is commutative with G_{m-r} and commutative with no other operators, has been included in (17). It is also evident that if the p-isomorphism t arises from a different arrangement of the group G_{m+1} it has not been included in the total number of p-isomorphisms given by (17), viz., all the p-isomorphisms given by (17) are distinct. Thus the truth of (17) has been established as was to be proved.

Since the formula given by (17) holds for $r = 1, 2, 3, \cdots, m-1$ and with the number of subgroups of order p^{m-r} in G_{m+1} we have completely established by induction that if (15) is true for $m = m$ then it is also true for $m = m + 1$. As we have previously established the truth of the formula for $m = 2$, 3, and 4, we have the following theorems:

THEOREM III. *The total number of p-isomorphisms, commutative with a given subgroup of order p^{m-r} and commutative with no other operators of G, in the group of isomorphisms of an abelian group G of order p^m and of type* $(1, 1, 1, \cdots)$ *is*

$$(p^{m-1} - 1)(p^{m-1} - p)(p^{m-1} - p^2) \cdots (p^{m-1} - p^{r-1}).$$

THEOREM IV. *The total number of p-isomorphisms in the group of isomorphisms of an abelian group G of order p^m and of type* $(1, 1, 1, \cdots)$ *is*

$$\sum_{r=1}^{m-1} (p^{m-1} - 1)(p^{m-1} - p) \cdots (p^{m-1} - p^{r-1}) \frac{(p^m - 1) \cdots (p^{m-r+1} - 1)}{(p^r - 1) \cdots (p - 1)}.$$

We shall now determine the total number of operators, whose orders are a power of p, in the group of isomorphisms of any abelian group G of order p^m.

We shall first consider the p-isomorphisms which are commutative with a subgroup of order p^{m-1}. Under these automorphisms of G an operator in the major co-set $G_{m-1}s_m$ can be made to correspond to itself multiplied by any operator in G_{m-1}, except the identity. This number is evidently

$$p^{m-1} - 1,$$

and the total number of p-isomorphisms which are commutative with p^{m-1} operators is the product of the number of p-isomorphisms commutative with a given subgroup of order p^{m-1} and the total number of such subgroups in G. Each of these subgroups must include the subgroup $G_{m-\lambda_1}$ and hence the total number of such subgroups is the number of subgroups of index p in the quotient group of G with respect to $G_{m-\lambda_1}$. This quotient group is of order p^{λ_1} and of type $(1, 1, 1, \cdots)$. Thus we have that the total number of

p-isomorphisms in the I of G commutative with exactly p^{m-1} operators is given by the following:

$$(p^{m-1} - 1)\frac{(p^{\lambda_1} - 1)}{(p - 1)}.$$

Since the automorphism of the independent generators of G completely establishes the automorphism of the group, and from the fact that all the operators of $G_{m_i} - G_{m_i - \lambda_i}$ $(i = 1, 2, 3, \cdots, n)$ are conjugate under I, we have by theorem I the total number of p-isomorphisms commutative with a given subgroup. Combining this with the number of such subgroups we have the total number of p-isomorphisms arising from the operators of $G_{m_i} - G_{m_i - \lambda_i}$ being commutative with the remaining independent generators, to be

$$\sum_{r=1}^{\lambda_i} (p^{m_i-1} - 1)(p^{m_i-1} - p) \cdots (p^{m_i-1} - p^{r-1}) \frac{(p^{\lambda_i} - 1) \cdots (p^{\lambda_i - r+1} - 1)}{(p^r - 1) \cdots (p - 1)}. \tag{18}$$

If we include the identity in this summation, then the total number of p-isomorphisms, including the identity, in the group of isomorphisms of G will be the product of these summations for $(i = 1, 2, \cdots, n)$.

Let us consider the summation

$$\sum_{r=1}^{b} (p^a - 1)(p^a - p) \cdots (p^a - p^{r-1}) \frac{(p^b - 1) \cdots (p^{b-r+1} - 1)}{(p^r - 1) \cdots (p - 1)} \quad (a \geqq b). \tag{19}$$

For small values of b this summation reduces to $p^{ab} - 1$. We shall show by induction that this is true for any value of b less than or equal to a. Suppose

$$\sum_{r=1}^{b} (p^a - 1)(p^a - p) \cdots (p^a - p^{r-1}) \frac{(p^b - 1) \cdots (p^{b-r+1} - 1)}{(p^r - 1) \cdots (p - 1)} = p^{ab} - 1, \tag{20}$$

and suppose that for $b = b + 1$ we have

$$\sum_{r=1}^{b+1} (p^a - 1)(p^a - p) \cdots (p^a - p^{r-1}) \frac{(p^{b+1} - 1) \cdots (p^{b-r+2} - 1)}{(p^r - 1) \cdots (p - 1)} = p^{a(b+1)} - 1. \tag{21}$$

If from the $(r + 1)$th term of (21) we subtract the product of p^a and the

rth term of (20), we have

$$(p^a-1)\frac{(p^{b+1}-1)}{(p-1)}+\sum_{r=1}^{b}(p^a-1)$$

$$\cdots(p^a-p^{r-1})\frac{(p^b-1)\cdots(p^{b-r+1}-1)}{(r^r-1)\cdots(p-1)} \tag{22}$$

$$\times\left[(p^a-p^r)\frac{(p^{b+1}-1)}{(p^{r+1}-1)}-p^a\right]=p^a-1.$$

For brevity we shall designate the rth term of (20) by U_r ($r = 1, 2, 3, \cdots, b$). With this we have the last term of (22) to be

$$-p^bU_b,$$

and the sum of the last two terms of (22) is

$$U_{b-1}\left[(p^a-p^{b-1})\frac{(p^{b+1}-1)}{(p^b-1)}-p^a-\frac{p^b(p^a-p^{b-1})(p-1)}{(p^b-1)}\right]$$

$$=-p^{b-1}U_{b-1}.$$

If to this we add the third from the last term, and etc., we shall assume the sum of the last $b-r+1$ terms to be equal to

$$-p^rU_r,$$

and if to this sum we add the $b-r+2$th term from the last, we have

$$U_{r-1}\left[(p^a-p^{r-1})\frac{(p^{b+1}-1)}{(p^r-1)}-p^a-\frac{p^r(p^a-p^{r-1})(p^{b-r+1}-1)}{(p^r-1)}\right]$$

$$=-p^{r-1}U_{r-1}.$$

Thus by induction we have the summation in (22) to be

$$-pU_1=-\frac{p(p^a-1)(p^b-1)}{(p-1)},$$

and hence

$$(p^a-1)\frac{(p^{b+1}-1)}{(p-1)}-\frac{p(p^a-1)(p^b-1)}{(p-1)}=p^a-1,$$

as was to be proved.

From (18) and (20) we have

$$1+\sum_{r=1}^{\lambda_i}(p^{m_i-1}-1)(p^{m_i-1}-p)$$

$$\cdots(p^{m_i-1}-p^{r-1})\frac{(p^{\lambda_i}-1)\cdot(p^{\lambda_i-r+1}-1)}{(p^r-1)\cdots(p-1)}=p^{(m_i-1)\lambda_i},$$

and hence the following theorem:

THEOREM V. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order p^{α_2}, $\cdots$, λ_n of order p^{α_n} ($\alpha_1>\alpha_2>\cdots>\alpha_n>0$), then the total number of p-iso-*

morphisms in the group of isomorphisms of G, including the identity, is

$$p_1^{\sum_1^n (m_i-1)\lambda_i}.$$

In case *the group G of order* p^m *is of type* $(1, 1, 1, \cdots)$, *we have the unique result that the total number of p-isomorphisms, including the identity, in the group of isomorphisms of G is the square of the order of a Sylow subgroup whose order is a power of p in the group of isomorphisms.*

From (6) and theorem V we have a new proof of the theorem* that a necessary and sufficient condition that the group of isomorphisms of an abelian group G of order p^m contains but one Sylow subgroup S of order p^M is that G does not contain two equal invariants.

The statement in the *Transactions of the American Mathematical Society,* Vol. 12, p. 397, that a necessary and sufficient condition that the group of isomorphisms of a group of order p^m involves only one Sylow subgroup of order p^M is that this group of order p^m involves a characteristic subgroup of order p^γ, for every value of γ from 1 to $m - 1$ is incorrect. The condition is necessary but not sufficient as may be seen in the case G is of type $(m - 2, 1, 1)$ discussed in 3.

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 36, p. 47.

THE ISODYADIC QUINTIC.

By J. C. Glashan.

Composition.—If the roots of the quintic

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0 \tag{1}$$

are expressible by radicals they will be of the form

$$\omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4, \tag{2}$$

in which

$$(\omega^5 - 1)/(\omega - 1) = 0, \qquad n = 1, 2, 3, 4, 5.$$

Also $(y_1y_4 - y_3y_2)^2$ must be rational and satisfy the relation

$$\begin{aligned}\{3125(y_1y_4 - y_3y_2)^6 - 2500A(y_1y_4 - y_3y_2)^4 \\ + 400B(y_1y_4 - y_3y_2)^2 - 64C\}^2 \\ - 1024(J^2 - 128K)(y_1y_4 - y_3y_2)^2 = 0,\end{aligned} \tag{3}$$

in which

$A = 3c^2 + e,$
$B = 15c^4 - 2c^2e + 8cd^2 - 2df + 3e^2,$
$C = (5c^3 - 3ce + 4d^2)(5c^3 - 4ce + 4d^2) + (c^2 + e)(2df - e^2) - cf^2,$
$J =$ the invariant of the fourth degree in the coefficients,
and $K =$ the invariant of the eighth degree.

(*See* Am. J. of M., Vol. XXIII, pp. 49 and 56.)

The quintic is isodyadic if

$$y_1y_4 = y_3y_2 \neq 0, \tag{4}$$

and ∴ if

$$c \neq 0 \qquad \text{and} \qquad C = 0. \tag{5}$$

$cC = 0$ may be arranged in the form

$$\{cf - d(c^2 + e)\}^2 - (c^3 - ce + d^2)\{(5c^2 - e)^2 + 16cd^2\} = 0. \tag{6}$$

Substituting $-p$ for c, $-p\alpha$ for $2d$, $-p\beta$ for e and $-p\gamma$ for f, this becomes

$$\{2\gamma - \alpha(\beta - p)\}^2 - \{\alpha^2 - 4(\beta + p)\}\{(\beta + 5p)^2 - 4p\alpha^2\} = 0. \tag{7}$$

Let, now,

$$(\beta + 5p)^2 - 4p\alpha^2 = \mu^2\{\alpha^2 - 4(\beta + p)\}; \tag{8}$$

then will

$$2\gamma - \alpha(\beta - p) = \mu\{4(\beta + p) - \alpha^2\} \tag{9}$$

and (8) may be written

$$(\beta + 5p + 2\mu^2)^2 = (4p + \mu^2)(\alpha^2 + 4\mu^2). \tag{10}$$

If now

$$4p + \mu^2 = \lambda^2(\alpha^2 + 4\mu^2) \tag{11}$$

(10) becomes

$$\beta + 5p + 2\mu^2 = \lambda(\alpha^2 + 4\mu^2). \tag{12}$$

Hence, α, λ, μ being any given numbers, if

$$4p = \lambda^2(4\mu^2 + \alpha^2) - \mu^2, \tag{11}'$$

$$\beta = \lambda(\alpha^2 + 4\mu^2) - 5p - 2\mu^2, \tag{12}'$$

$$2\gamma = \beta(\alpha + 4\mu) - p(\alpha - 4\mu) - \alpha^2\mu, \tag{9}'$$

the quintic

$$x^5 - 5p(2x^3 + \alpha x^2 + \beta x) - p\gamma = 0 \tag{13}$$

will be isodyadic.

Examples.—1. If $\alpha = -19$, $\lambda = \frac{9}{13}$, $\mu = 11$, the quintic is

$$x^5 - 355(2x^3 - 19x^2 - 12x) - 7171 = 0.$$

2. If $\alpha = -19$, $\lambda = -\frac{9}{13}$, $\mu = 11$, the quintic is

$$x^5 - 355(2x^3 - 19x^2 - 1182x) + 1031204 = 0.$$

(The former of these is the Gaussian of the quintic section of $x^{71} - 1 = 0$; the latter is its quadratic conjugate.)

3. If $\alpha = -29$, $\lambda = \frac{57}{109}$, $\mu = 121$, the quintic is

$$x^5 - 2005(2x^3 - 29x^2 - 222x) - 589871 = 0.$$

4. If $\alpha = -29$, $\lambda = -\frac{57}{109}$, $\mu = 121$, the quintic is

$$x^5 - 2005(2x^3 - 29x^2 - 62352x) + 4404222699 = 0.$$

(Ex. 3 is the Gaussian of the quintic section of $x^{401} - 1$; Ex. 4 is its quadratic conjugate.)

Solution.—If the coefficients of the quintic

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0 \tag{1}$$

satisfy the relations $c \neq 0$ and

$$(5c^3 - 3ce + 4d^2)(5c^3 - 4ce + 4d^2) + (c^2 + e)(2df - e^2) - cf^2 = 0,$$

it is isodyadic and may be solved as follows.

Write (1) in the form

$$x^5 - 5p(2x^3 + \alpha x^2 + \beta x) - p\gamma = 0; \tag{13}$$

then will the roots be

$$x_n = \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4,$$

in which

$$(\omega^5 - 1)/(\omega - 1) = 0, \qquad n = 1, 2, 3, 4, 5;$$

and

$$y_1y_4 = y_3y_2 = p, \tag{i}$$
$$y_1^2y_3 + y_4^2y_2 + y_3^2y_4 + y_2^2y_1 = p\alpha, \tag{ii}$$
$$y_1^3y_2 + y_4^3y_3 + y_3^3y_1 + y_2^3y_4 = p(p + \beta), \tag{iii}$$
$$(y_1^5 + y_4^5 + y_3^5 + y_2^5 = p\gamma). \tag{iv}$$

For convenience of notation write

$$\rho = \alpha^2 - 4(p + \beta), \tag{v}$$
$$\sigma = \alpha + \sqrt{\rho},$$
$$\tau = \alpha - \sqrt{\rho},$$
$$\mu = \{\alpha(\beta - p) - 2\gamma\}/\rho,$$
$$\therefore \quad \sigma\tau = 4(p + \beta)$$
$$\text{and} \quad \sqrt{\{(\sigma^2 - 16p)(\tau^2 - 16p)\}} = 4\mu\sqrt{\rho}.$$

From $\sqrt{\{(\text{ii})^2 - 4(\text{iii})\}}$ and (v)

$$y_1^2y_3 + y_4^2y_2 - y_3^2y_4 - y_2^2y_1 = p\sqrt{\rho};$$
$$\therefore \quad y_1^2y_3 + y_4^2y_2 = \tfrac{1}{2}p\sigma$$
$$\text{and} \quad y_3^2y_4 + y_2^2y_1 = \tfrac{1}{2}p\tau;$$
$$\therefore \quad y_1^2y_3 - y_4^2y_2 = \tfrac{1}{2}p\sqrt{(\sigma^2 - 16p)}$$
$$\text{and} \quad y_3^2y_4 - y_2^2y_1 = \tfrac{1}{2}p\sqrt{(\tau^2 - 16p)};$$
$$\therefore \quad y_1^2y_3 = \tfrac{1}{4}p\{\sigma + \sqrt{(\sigma^2 - 16p)}\},$$
$$y_4^2y_2 = \tfrac{1}{4}p\{\sigma - \sqrt{(\sigma^2 - 16p)}\},$$
$$y_3^2y_4 = \tfrac{1}{4}p\{\tau + \sqrt{(\tau^2 - 16p)}\},$$
$$y_2^2y_1 = \tfrac{1}{4}p\{\tau - \sqrt{(\tau^2 - 16p)}\},$$
$$\therefore \quad y_1^5 = \tfrac{1}{64}p\{\sigma + \sqrt{(\sigma^2 - 16p)}\}^2\{\tau - \sqrt{(\tau^2 - 16p)}\}$$
$$= \tfrac{1}{8}p\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho} + \sqrt{[\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\},$$
$$y_4^5 = \tfrac{1}{8}p\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho} - \sqrt{[\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\},$$
$$y_3^5 = \tfrac{1}{8}p\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho} + \sqrt{[\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\},$$
$$y_2^5 = \tfrac{1}{8}p\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho} - \sqrt{[\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\}.$$

Expressed in the coefficients of (1) these become

$$y_1^5 = -\tfrac{1}{4}[f + R - \sqrt{\{(f + R)^2 + 16c^5\}}],$$
$$y_4^5 = -\tfrac{1}{4}[f + R + \sqrt{\{(f + R)^2 + 16c^5\}}],$$
$$y_3^5 = -\tfrac{1}{4}[f - R - \sqrt{\{(f - R)^2 + 16c^5\}}],$$
$$y_2^5 = -\tfrac{1}{4}[f - R + \sqrt{\{(f - R)^2 + 16c^5\}}],$$

in which

$$R = \frac{4(c^2 - e)(3c^2 - e) + cd^2 + df}{\sqrt{\{16c(c^2 - e) + d^2\}}}.$$

Examples.—1.

$$x^5 - 55(2x^3 + x^2 - 42x) + 979 = 0.$$

Here, $p = 11$, $\alpha = 1$, $\beta = -42$, $\gamma = -89$ and $\therefore \rho = 125$, $\mu = 1$,

$$y_1y_4 = y_3y_2 = 11,$$
$$y_1^2y_3 + y_4^2y_2 + y_3^2y_4 + y_2^2y_1 = 11,$$
$$y_1^3y_2 + y_4^3y_3 + y_3^3y_1 + y_2^3y_4 = -341;$$
$$\therefore \quad y_1^2y_3 + y_4^2y_2 = \tfrac{11}{2}(1 + 5\sqrt{5}),$$
$$y_3^2y_4 + y_2^2y_1 = \tfrac{11}{2}(1 - 5\sqrt{5});$$
$$\therefore \quad y_1^2y_3 = \tfrac{11}{4}\{1 + 5\sqrt{5} + \sqrt{(-50 + 10\sqrt{5})}\},$$
$$y_2^2y_1 = \tfrac{11}{4}\{1 - 5\sqrt{5} - \sqrt{(-50 - 10\sqrt{5})}\};$$
$$\therefore \quad y_1^5 = -\tfrac{11}{4}[89 + 25\sqrt{5} - \sqrt{\{(89 + 25\sqrt{5})^2 - 16 \times 11^3\}}].$$

(This is the cyclotomic quintic for $x^{11} - 1 = 0$.)

2. The quadratic conjugate of Ex. 1 is

$$x^5 - 55(2x^3 + x^2 - 72x) + 1804 = 0$$

with $\quad y_1 = -11[41 + 25\sqrt{5} - \sqrt{\{(41 + 25\sqrt{5})^2 - 11^3\}}].$

3. $\quad x^5 - 800(2x^3 + 5x^2 - 534x) - 160 \times 2828 = 0.$

Here $\quad \rho = 39^2 \quad$ and $\quad \mu = -6,$

$$y_1^5 = 1600(59 + 31i\sqrt{39}),$$
$$y_3^5 = 1280(103 + 37i\sqrt{39}).$$

A. If (13) be cyclotomic and p be a prime number $\equiv 1 \bmod 10$, α will be $\equiv 1 \bmod 5$ and $5\{(\beta + 5p)^2 - 4p\alpha^2\}$ will be a rational square (AM. J. M., Vol. XXI, p. 272). Hence,

If p is a prime $\equiv 1 \bmod 10$, $4p = u^2 - 5v^2$ is solvable in integers.

B. In general (AM. J. M., Vol. XXI, p. 277), if n and p are primes, $n \equiv 1 \bmod 4$, $p \equiv 1 \bmod 2n$, $4p = u^2 - nv^2$ is solvable in integers. Hence, also, if $n \equiv 1 \bmod 8$, $p = u^2 - nv^2$ is solvable, but if $n \equiv 5 \bmod 8$, $p = u^2 - 5v^2$ is solvable only if $4 = u^2 - 5v^2$ is solvable; e.g., $p = u^2 - 5v^2$ has no integral solutions if $n \equiv 1 \mod 4(2m + 3)^2$. Ex. $773 = 7205048979^2 - 193 \times 518630774^2$ and $4 \times 2153 = 71821^2 - 269 \times 4379^2$.

ON CLASS NUMBER RELATIONS FOR BILINEAR FORMS IN FOUR VARIABLES.

By E. T. Bell.

1. In the notation of Kronecker,* $\overline{Cl}(\Delta)$ is the number of classes of bilinear forms

$$Ax_1y_1 + Bx_1y_2 - Cx_2y_1 + Dx_2y_2, \qquad \Delta \equiv AD + BC,$$

of determinant Δ, in which at least one of the extreme coefficients A, D is odd and $B + C$ is even. He shows (loc. cit., p. 494) that

$$\sum_{n=1}^{\infty} \overline{Cl}(n)q^n = \frac{6k^2K^2}{\pi^2 \sqrt[4]{q}} \int_0^1 H((2v+1)K) \sin^2 am2vK \cos v\pi dv,$$

in the notation of the *Fundamenta Nova*. In other words that part of the Fourier cosine series for the function

$$\frac{6k^2K^2}{\pi^2 \sqrt[4]{q}} H((2v+1)K) \operatorname{sn}^2 2vK \cos v\pi,$$

which is independent of the argument v is $\sum \overline{Cl}(n)q^n$. The part containing v is not obtained, and is in fact neglected in the entire discussion. Attending to this part we find several interesting class number relations for binary quadratic forms and, as the simplest consequence of the whole expansion, an elegant relation of a similar kind for $\overline{Cl}(n)$. Here we shall restrict the discussion to this relation, believed to be the first of its kind. We recall that Kronecker obtained $\overline{Cl}(n)$ in the form (loc. cit., p. 452) $12\sum \mathrm{F}(n - h^2)$, the $\sum$ referring to all integers $h \gtrless 0$ such that $n - h^2 > 0$, $\mathrm{F}(n)$ being his class number function† for binary quadratic forms of negative determinant $-n$. This will afford a check on one phase of the following developments.

2. Changing Kronecker's notation to the small thetas of Jacobi, we are to expand $\psi(v)$ in a cosine series, where

$$\sqrt[4]{q}\psi(v) \equiv \frac{3}{2} \cos v\vartheta_2^2\vartheta_3^2 \frac{\vartheta_1^2(v)\vartheta_2(v)}{\vartheta_0^2(v)}.$$

Denote by $\psi_0(v)$, $\psi_1(v)$ respectively the parts of $\psi(v)$ independent of v, not independent of v. Then $\psi_0(v) = \sum \overline{Cl}(n)q^n$. To obtain $\psi_1(v)$ we introduce $\chi(v)$, defined by

$$\chi(v) \equiv \vartheta_2^2\vartheta_3 \frac{\vartheta_1^2(v)\vartheta_2(v)}{\vartheta_0^2(v)},$$

* Werke, Vol. 2, pp. 425–495. See also Dickson's "History," Vol. 3, pp. 129–130, 286.

† For the distinction between F and the more usual F see Dickson's "History," Vol. 3, p. 109; or Kronecker, loc. cit., p. 449.

which, in another notation, is one of the functions discussed by G. Humbert.* The reduced form of $\chi(v)$ suitable for this kind of work was given in a former paper.† We found

$$\chi(v) = 2\sum q^{\alpha/4}\left[2\sum F(\alpha - b^2)\cos bx - \sum'(\delta - d)\cos\left(\frac{\delta + d}{2}\right)v\right],$$

in which the outer $\sum$ refers to all positive integers $\alpha \equiv 1 \bmod 4$, the coefficient of $q^{\alpha/4}$ is in $[\,]$, the inner $\sum$ refers to all odd integers $b \gtreqless 0$ which render $\alpha - b^2 > 0$, $F(n)$ is the class number for binary quadratic forms of determinant $-n$ with the usual conventions, and $\sum'$ refers to all positive divisors d, δ of α such that $\alpha = d\delta$, $d > \sqrt{\alpha}$. We have

$$\sqrt[4]{q}\psi(v) = \tfrac{3}{2}\cos v \cdot \vartheta_3 \cdot \chi(v).$$

Writing $c = |b|$ we find

$$\chi(v)\cos v = 4\sum q^{\alpha/4}[\sum F(\alpha - c^2)\{\cos(c+1)v + \cos(c-1)v\}]$$
$$+ \sum q^{\alpha/4}\left[\sum'(d-\delta)\left\{\cos\left(\frac{\delta+d}{2}+1\right)v + \cos\left(\frac{\delta+d}{2}-1\right)v\right\}\right],$$

the sum with respect to c extending to all $c = 1, 3, 5, \cdots$, such that

$$\alpha - c^2 > 0.$$

Calculating the part of $\chi(v)\cos v$ independent of v, we note that zero arguments can appear only in terms of the form $\cos(c-1)v$, $\cos\left(\frac{\delta+d}{2} - 1\right)v$. From the first, $c = 1$; from the second $\delta + d = 2$, and hence $\delta = d = 1$. But the restriction $d < \sqrt{\alpha}$ imposed by $\sum'$ excludes the second. The part of $\chi(v)\cos v$ independent of v is therefore $4\sum q^{\alpha/4}F(\alpha - 1)$. Hence from the definitions of the functions,

$$\psi_0(v) = 6\sum q^{(\alpha-1)/4}F(\alpha - 1) \times \sum q^{n^2},$$

the second $\sum$ referring to $n = 0, \pm 1, \pm 2, \cdots$. Recalling that $\alpha \equiv 1 \bmod 4$, and that $F(4n) = 2F(n)$ for all integers $n \geqq 0$, we now replace α by $4n + 1$ and get

$$\psi_0(v) = 12\sum q^n[\sum F(n - h^2)] = \sum q^n \bar{C}l(n),$$

the sum with respect to h referring to all $h = 0, \pm 1, \pm 2, \cdots$, such that $n - h^2 > 0$. This checks $\psi_0(v)$.

3. The part of $\chi(v)\cos v$ not independent of v can now be written (on

* *Journal des Mathématiques*, 6 Série, Vol. 3 (1907), pp. 337–449. Humbert's entire set of 24 expansions, ibid., pp. 349–354, can be used to derive bilinear class number relations.

† *Quarterly Journal*, Vol. 59, No. 196 (1923), pp. 322–337, where the entire set of 24 is reduced.

omitting the term $\cos(c-1)v$ for $c=1$) in the form

$$4\Sigma q^{\alpha/4}[\Sigma\{F(\alpha-\overline{s-1^2})+F(\alpha-\overline{s+1^2})\}\cos sx]$$
$$-\Sigma q^{\alpha/4}\left[\Sigma'(\delta-d)\left\{\cos\left(\frac{\delta+d}{2}+1\right)v+\cos\left(\frac{\delta+d}{2}-1\right)v\right\}\right],$$

the inner Σ in the first referring to all even integers $s>0$ such that no F occurs with a zero or negative argument. Hence, multiplying $\chi(v)\cos v$ by ϑ_3, dividing the result by $\sqrt[4]{q}$, and making some obvious simplifications in the resulting series, we finally obtain

$$\psi(v)=\Sigma q^n[\overline{C}l(n)+24\Sigma\Sigma\{F(n-t_1^2-t^2+t)$$
$$+F(n-t_1^2-t^2-t)\}\cos 2tv]$$
$$-\frac{3}{2}\Sigma q^n\left[\Sigma'(\delta_1-d_1)\left\{\cos\left(\frac{\delta_1+d_1}{2}+1\right)v\right.\right.$$
$$\left.\left.+\cos\left(\frac{\delta_1+d_1}{2}-1\right)v\right\}\right],$$

in which the summations are as follows: the outer Σ's refer to $n=1, 2, 3, \cdots$; the $\Sigma\Sigma$ to all $t_1 \gtreqless 0$ and to all $t>0$ such that the argument of no F is zero or negative; the Σ' to all divisors d_1, δ_1 of $\alpha_1=d_1\delta_1$ such that $d_1<\sqrt{\alpha_1}$, and α_1 is determined from *all* solutions of $0<4n+1-4t_1^2=\alpha_1$ (n fixed, t_1 as above).

Note that the second sum contains no term independent of v. The first of the sums in $\psi(v)$ can be materially simplified, at least in expression. For a moment write $X(n)=24\Sigma\Sigma F(n-t_1^2-t^2+t)$, this being one of the double sums above. Then, the summation referring to all values of $n_1=1, 2, 3, \cdots$, and of n, t_1, t as defined, we have

$$\begin{aligned}\Sigma X(n)q^n &\equiv 24\Sigma q^n[\Sigma\Sigma F(n-t_1^2-t^2+t)],\\ &=24\Sigma q^{n_1}F(n_1)\times\Sigma q^{t_1^2}\times\Sigma q^{t^2-t},\\ &=24\Sigma q^n[\Sigma F(n-t_1^2)]\times\Sigma q^{t^2-t},\\ &=2\Sigma q^n[\Sigma\overline{C}l(n-t^2+t)],\end{aligned}$$

the inner Σ referring to all $t>0$ such that $n-t^2+t>0$. Hence

$$\psi(v)=\Sigma q^n[\overline{C}l(n)+2\Sigma\overline{C}l(n-a^2+a)\cos 2av]$$
$$-\frac{3}{2}\Sigma q^n\left[\Sigma'(\delta_1-d_1)\left\{\cos\left(\frac{\delta_1+d_1}{2}+1\right)v\right.\right.$$
$$\left.\left.+\cos\left(\frac{\delta_1+d_1}{2}-1\right)v\right\}\right],$$

the Σ with respect to a referring to all $a=\pm 1, \pm 2, \pm 3, \cdots$ such that $n-a^2+a>0$. The rest of the letters are as before.

4. As the simplest consequence of this expansion we note that $\psi(0) = 0$, since $\vartheta_1(0) = 0$. Hence

$$\bar{C}l(n) + 2\sum\bar{C}l(n - a^2 + a) = 3\sum'(\delta_1 - d_1),$$

the notation being as before. Denote by $\lambda(n)$ the excess of the sum of all those divisors of n that exceed $\sqrt{n}$ over the sum of all those that are exceeded by $\sqrt{n}$. Then $\sum'(\delta_1 - d_1) = \lambda(\alpha_1)$, and hence the right of the foregoing equation is $3\sum\lambda(4n + 1 - 4t^2)$, from the definition of α_1. Finally then we obtain the bilinear class number relation

$$\bar{C}l(n) + 2\sum\bar{C}l(n - a^2 + a) = 3[\lambda(4n + 1) + \sum\lambda(4n + 1 - 4a^2)],$$

the summations referring to all $a = \pm 1, \pm 2, \pm 3, \cdots$ that render the arguments of the function > 0.

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RELATIVE INCLUSIVENESS OF CERTAIN DEFINITIONS OF SUMMABILITY.

By David Sherman Morse.

1. Introduction.

The number of definitions that have been given for evaluating a divergent series is so large that the study of the relationships existing between various definitions has become of importance. It is the purpose of this paper to contribute to this study by considering certain definitions from the standpoint of relative inclusiveness.

The terminology used in the literature of the subject differs somewhat with different writers. Because of this, it seems desirable to define some of the terms we are going to use.* A definition is said to be *regular* if it evaluates every convergent sequence, giving to it the value to which it converges. A definition, A, is said to *include* another, B, provided every sequence summable B is summable A to the same value. Two definitions are said to be *equivalent* if each includes the other. Two definitions are said to be *mutually consistent* if, whenever each of them evaluates a sequence, the two values are the same.

We shall have occasion to refer to several definitions that assign sums to divergent series. We now list some of these definitions, together with the symbols by which we refer to them. Unless otherwise stated, we use the symbol

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$$

and define

$$x_n = u_1 + u_2 + \cdots + u_n.$$

Cesàro's rth Mean, C_r.†

This definition assigns to a divergent series the value $\lim_{n\to\infty} y_n$, if this limit exists, where

$$y_n = \frac{S_n^{(r)}}{A_n^{(r)}};$$

* The terminology here used is that of Prof. Hurwitz's paper, *Report on topics in the theory of divergent series;* Bulletin of the American Mathematical Society, Vol. 28 (1922), p. 17. This paper will be referred to as *Report.*

† *Bulletin des Sciences Mathématiques*, Ser. 2, Vol. 14 (1880), p. 114.

with

$$S_n^{(r)} = \frac{r(r+1)\cdots(r+n-2)}{(n-1)!}x_1 + \frac{r(r+1)\cdots(r+n-3)}{(n-2)!}x_2 + \cdots + rx_{n-1} + x_n,$$

and

$$A_n^{(r)} = 1 + r + \frac{r(r+1)}{2!} + \cdots + \frac{r(r+1)\cdots(r+n-2)}{(n-1)!}.$$

An equivalent expression for y_n is:

$$y_n = \frac{n(n+1)\cdots(n+r-2)x_1 + \cdots + (r-1)!\,x_n}{n(n+1)\cdots(n+r-1)/r}.$$

*LeRoy's Definition.**

The generalized value for a divergent series according to this definition is $\lim_{t\to 1-} y(t)$ where

$$y(t) = \sum_{n=1}^{\infty} \frac{\Gamma(\overline{n-1}\,t+1)}{(n-1)!}u_n, \qquad 0 < t < 1.$$

Borel's Integral Definition.†

Form the function

$$U(t) = u_1 + u_2 t + u_3 t^2/2! + \cdots;$$

if $U(t)$ converges for all t, and if

$$\int_0^{\infty} e^{-t}U(t)dt$$

exists and is equal to l, then the series is said to be summable Borel to the value l.

The Exponential Mean, E_r.‡

The generalized value is $\lim_{n\to\infty} y_n$, if this limit exists, where

$$y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!}\frac{(r-1)^{n-k}}{r^{n-1}}x_k.$$

The Riesz Means, (R_λ, k).§

Form

$$y(t) = t^{-k}\sum_{\lambda_n < t}(t-\lambda_n)^k u_n,$$

* *Annales Fac. Sci. Toulouse*, Ser. 2, Vol. 2 (1900), p. 327.

† Borel: *Leçons sur les séries divergentes* (Paris, 1901).

‡ Hausdorff: *Mathematische Zeitschrift*, Vol. 9 (1921), p. 86. Also, Hurwitz: *Report.*

§ *Comptes Rendus*, Vol. 149 (1909), p. 910.

in which λ_n is a sequence of positive real increasing numbers whose limit is infinite. Then the generalized value of the series is given by

$$\lim_{t \to \infty} y(t).$$

The Dirichlet's Series Definitions, D_{λ_n}.

Hardy* gives a definition which he states is "substantially equivalent" to LeRoy's and has, in the case of a power series of finite radius of convergence, "precisely similar powers." This definition is†

$$\lim_{\delta \to 0} \sum_{\nu=0}^{\infty} u_\nu e^{-\delta \nu \log \nu}.$$

This definition leads us to a set of definitions which we shall call the Dirichlet's series definitions.

We define as the generalized value of the series, $\sum_{n=1}^{\infty} u_n$,

$$\lim_{\substack{t \to t_0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-\lambda_n t},$$

where T is a point set, in the real or complex plane, having a limit point t_0, not belonging to the set, and where λ_n is a sequence of positive increasing real numbers whose limit is infinite.

A special case of the Dirichlet's series definitions is well known; namely, the case where $\lambda_n = n$, $t_0 = 0$, and T is a point set along the axis of reals, $0 < t$. This gives

$$\lim_{t \to 0} \sum_{n=1}^{\infty} u_n e^{-nt} = \lim_{t \to 0} e^{-t} \sum_{n=1}^{\infty} u_n (e^{-t})^{n-1} = \lim_{t \to 0} \sum_{n=1}^{\infty} u_n (e^{-t})^{n-1}.$$

If we replace e^{-t} by x, we get

The Euler Power Series Definition.

The generalized value is given by

$$\lim_{x \to 1-} \sum_{n=1}^{\infty} u_n x^{n-1}.$$

We wish to study another special case of this set of definitions which differs from Hardy's definition only to the extent that we write $\sum_{n=1}^{\infty} u_n$ while Hardy wrote $\sum_{n=0}^{\infty} u_n$. This definition, we call

* *Quarterly Journal of Mathematics*, Vol. 42 (1911), p. 193.

† $\delta \to \infty$, in above reference, is evidently a typographical error.

The Dirichlet's Series Definition $D_{n \log n}$.

$$\lim_{\substack{t \to 0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-nt \log n}.$$

This definition is regular* when T is a set of points along the axis of reals. Sufficient conditions that a definition of the kind be regular are given by Carmichael.† They are, in this case,

1. $\lim_{t \to 0} e^{-nt \log n} = 1$.
2. $\sum_{n=1}^{\infty} \left| e^{-nt \log n} - e^{-(n+1)t \log (n+1)} \right|$ converges for each $t > 0$ and is bounded for all $t > 0$.

The first condition requires no proof. The second condition is satisfied also. Since $n \log n$ is an increasing function for $n > 1/e$,

$$e^{-nt \log n} - e^{-(n+1)t \log (n+1)} > 0$$

for $n > 1/e$, and the sum in condition 2 can be replaced by

$$\sum_{n=1}^{\infty} \left[e^{-nt \log n} - e^{-(n+1)t \log (n+1)} \right].$$

Since $\lim_{n \to \infty} e^{-nt \log n} = 0$ for $t > 0$, it follows that

$$\sum_{n=1}^{\infty} \left[e^{-nt \log n} - e^{-(n+1)t \log (n+1)} \right] = 1$$

and the second condition is satisfied.

In section 2 of this paper, we prove that the Dirichlet's series definition, $D_{n \log n}$, includes C_r. Then, in section 3, we consider this definition, $D_{n \log n}$, from a Dirichlet's series standpoint, giving a proof of regularity when $t \to 0$ over other point sets than the axis of reals. We also give another proof that $D_{n \log n}$ includes C_r.

There are, in the literature, at least two misleading statements regarding the relationship of the LeRoy and Borel definitions. Hardy, in the *Quarterly Journal of Mathematics*, Vol. 35, p. 37, states the following conclusion to a proof. "It follows that if $u_0 + u_1 + u_2 + \cdots$ is summable by the exponential method, it is summable also by M. LeRoy's method, and that the two sums are the same." This seems to imply that LeRoy's definition includes Borel's. Upon looking up a reference that the author makes to one of his earlier papers in order to justify a step in the proof, it becomes evident that he has proved the theorem only for the case where the series $u_0 + u_1 + u_2 + \cdots$ is convergent and has, consequently, proved that

* The definition is also *totally regular*. See *Report*, p. 30.

† *Bulletin of the American Mathematical Society*, Vol. 25 (1918–19), p. 120. These conditions are, in fact, necessary.

LeRoy's definition is regular provided Borel's is regular. If the section, in which the proof occurs, is read from the beginning, this is seen to be the purpose of the author. Bromwich also has made a misleading statement in this connection. In his "Theory of Infinite Series," page 299, he proves that, if a series is summable Borel, it is then summable LeRoy to the same value *provided* the series in the LeRoy definition converges absolutely. At the end of the proof, he forgets, evidently, his assumption of absolute convergence, and states: "Hence LeRoy's definition coincides with Borel's, whenever the latter is convergent."

In section 4, we show that Borel's definition is not included in LeRoy's. We prove that, if a series is summable Borel, then it is summable LeRoy to the same value provided the LeRoy series converges; and give a sufficient condition that a series summable Borel shall be summable LeRoy to the same value. From this condition, we get several theorems, of which the following is typical: *If a series is summable C_r and Borel, it is summable LeRoy to the same value.* We also prove, in this section, that the Dirichlet's series definition $D_{n \log n}$ does not include Borel's definition and that Borel's definition does not include $D_{n \log n}$.

In section 5, we have indicated how far we have been able to go towards proving that LeRoy's definition includes C_r and have pointed out why we believe that LeRoy's definition does include C_r.

2. The Relation between the Dirichlet's Series Definition $D_{n \log n}$ and C_r.

Bromwich* has proved that if a series, $\sum_{n=1}^{\infty} u_n$, is summable C_r to the sum l and if $f_n(t)$ is a function of t such that

1. $\lim_{t \to 0} f_n(t) = 1$,
2. $\lim_{n \to \infty} n^r f_n(t) = 0, \quad t > 0$,
3. $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} f_n(t)|$ converges for each $t > 0$ and is bounded for all $t > 0$;

then $\sum_{n=1}^{\infty} f_n(t) u_n$ converges, $t > 0$, and

$$\lim_{t \to 0} \sum_{n=1}^{\infty} f_n(t) u_n = l.\dagger$$

Hurwitz‡ has shown that (2) may be replaced by the more general condition

* *Mathematische Annalen*, Vol. 65 (1907–08), p. 359.

† Compare form of statement used by C. N. Moore, *Bulletin of American Mathematical Society*, Vol. 25, page 267.

‡ Abstract, *Bulletin of American Mathematical Society*, Vol. 28 (1922), page 156.

2′. for each $t > 0$, $n^r f_n(t)$ is bounded for all n (the bound may depend on t);

and that conditions (1), (2′), and (3) are necessary if the result is to hold for every series summable C_r.

We make use of this theorem to prove

THEOREM I. *The Dirichlet's series definition, $D_{n \log n}$, includes C_r for all positive integral values of r.*

We assume that $\sum_{n=1}^{\infty} u_n$ is summable C_r to the value l and desire to prove that

$$\sum_{n=1}^{\infty} e^{-nt \log n} u_n$$

converges, $t > 0$, and that

$$\lim_{t \to 0} \sum_{n=1}^{\infty} e^{-nt \log n} u_n = l.$$

From Bromwich's theorem, this will be true provided

1. $\lim_{t \to 0} e^{-nt \log n} = 1$,
2. $\lim_{t \to 0} n^r e^{-nt \log n} = 0, \quad t > 0$,
3. $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} e^{-nt \log n}|$ converges for each $t > 0$ and is bounded for all $t > 0$.

Before showing that these three conditions are satisfied, we give four lemmas.

LEMMA 1. *If*

1. $\sum_{k=p}^{q} F(k, t)$ *is bounded for all p, q and all* $t > 0$,
2. *the number of changes of sign (with varying k) of* $F(k, t)$ *is finite for each constant* $t > 0$ *and bounded for all* $t > 0$,

then

$$\sum_{k=1}^{\infty} |F(k, t)| \text{ converges for each } t > 0 \text{ and is bounded for all } t > 0.$$

In this lemma, condition 1 may be replaced by the equivalent condition

1′. $\sum_{k=1}^{q} F(k, t)$ is bounded for all q and all $t > 0$.

For each t there will exist an index $k = g(t)$ such that for $k > g(t)$, $F(k, t)$ has the same sign. Form $\sum_{k=1}^{n} |F(k, t)|$, where $n > g(t)$, and break this sum into partial sums each consisting of terms in which $F(k, t)$ retains

one sign. Then

$$\sum_{k=1}^{n} |F(k,t)| = \sum_{(1)} |F(k,t)| + \sum_{(2)} |F(k,t)| + \cdots + \sum_{(n)} |F(k,t)|$$
$$= |\sum_{(1)} F(k,t)| + |\sum_{(2)} F(k,t)| + \cdots + |\sum_{(n)} F(k,t)|.$$

By condition 1, each group is less than a constant A, independent of t; and by condition 2, the number of groups is less than a constant M, independent of t; hence

$$\sum_{k=1}^{n} |F(k,t)| < M \cdot A;$$

from which the conclusion of the theorem follows.

LEMMA 2. *If* $\varphi(x) = \dfrac{f(x)}{x}$, *then the rth derivative of* $\varphi(x)$ *is given by*

$$\varphi^{(r)}(x) = \frac{\sum_{k=0}^{r} (-1)^k \dfrac{r!}{(r-k)!} x^{r-k} f^{(r-k)}(x)}{x^{r+1}}.$$

LEMMA 3. *If* $\varphi(n)$ *possesses r derivatives, then*

$$(-1)^r \Delta^r \varphi(n) = \varphi^{(r)}[n + \theta_r], \qquad 0 < \theta_r < r.$$

LEMMA 4. *If* $f(x) = e^{-tx \log x}$, *then the rth derivative of* $f(x)$ *is given by*

$$f^{(r)}(x) = (-t)^r f(x) \Big[(1 + \log x)^r - \frac{r(r-1)}{2} \frac{(1+\log x)^{r-2}}{tx} \frac{\varphi_{r-3}(\log x)}{t^2 x^2}$$
$$+ \cdots + \frac{\varphi_1(\log x)}{t^{r-2} x^{r-2}} + \frac{(r-2)!}{t^{r-1} x^{r-1}} \Big],$$

where $\varphi_k(\log x)$ represents a polynomial in $\log x$ of degree $\leq k$.

This lemma can be established by mathematical induction.

We are now in a position to prove that the function $e^{-nt \log n}$ satisfies the three conditions of Bromwich's theorem.

The first condition that $\lim_{t \to 0} e^{-nt \log n} = 1$ is satisfied.

In order to show that the second condition is satisfied, we write

$$n^r e^{-nt \log n} = n^{r-nt} \to 0 \qquad \text{as} \qquad n \to \infty, \quad t > 0.$$

It remains to prove that $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} e^{-nt \log n}|$ converges for each $t > 0$ and is bounded for all $t > 0$. If n^r in this expression is replaced by any polynomial of the rth degree in n, the new expression will surely converge and be bounded, or fail to converge and be bounded, under the same conditions as the original expression. We, therefore, choose to show that

$$\sum_{n=1}^{\infty} \frac{(n+r)!}{n!} |\Delta^{r+1} e^{-nt \log n}|$$

converges for each $t > 0$ and is bounded for all $t > 0$. We do this by showing that the conditions of Lemma 1 are satisfied. We show that condition 1′ of the lemma is satisfied by computing

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{(k+r)!}{k!} \Delta^{r+1} f_k(t) &= \sum_{k=1}^{n-1} \frac{(k+r)!}{k!} f_k(t) \\ &- \sum_{k=1}^{n-1} (r+1) \frac{(k+r)!}{k!} f_{k+1}(t) + \sum_{k=1}^{n-1} \frac{(r+1)r}{2!} \frac{(k+r)!}{k!} f_{k+2}(t) \\ &- \cdots (-1)^{r+1} \sum_{k=1}^{n-1} \frac{(r+1)r}{2!} \frac{(k+r)!}{k!} f_{k+r-1}(t) \\ &+ (-1)^r \sum_{k=1}^{n-1} (r+1) \frac{(k+r)!}{k!} f_{k+r}(t) \\ &+ (-1)^{r+1} \sum_{k=1}^{n-1} \frac{(k+r)!}{k!} f_{k+r+1}(t),\end{aligned}$$

where $f_k(t)$ is written for $e^{-kt \log k}$.

We can write

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{(k+r)!}{k!} \Delta^{r+1} f_k(t) &= \sum_{k=1}^{n-1} \frac{(k+r)!}{k!} f_k(t) \\ &- \sum_{k=2}^{n} (r+1) \frac{(k+r-1)!}{(k-1)!} f_k(t) + \sum_{k=3}^{n+1} \frac{(r+1)r}{2!} \frac{(k+r-2)!}{(k-2)!} f_k(t) \\ &- \cdots (-1)^{r+1} \sum_{k=r}^{n+r-2} \frac{(r+1)r}{2!} \frac{(k+1)!}{(k-r+1)!} f_k(t) \\ &+ (-1)^r \sum_{k=r+1}^{n+r-1} (r+1) \frac{k!}{(k-r)!} f_k(t) + (-1)^{r+1} \sum_{k=r+2}^{n+r} \frac{(k-1)!}{(k-r-1)!} f_k(t).\end{aligned} \tag{1}$$

If $n \geq r+3$, we can write part of (1) as follows:

$$\begin{aligned}\sum_{k=r+2}^{n-1} \Bigg[&\frac{(k+r)!}{k!} - (r+1) \frac{(k+r-1)!}{(k-1)!} + \frac{(r+1)r}{2} \frac{(k+r-2)!}{(k-2)!} \\ &- \cdots (-1)^r (r+1) \frac{k!}{(k-r)!} + (-1)^{r+1} \frac{(k-1)!}{(k-r-1)!} \Bigg] f_k(t).\end{aligned} \tag{2}$$

The sum of the first $p+1$ terms inside the bracket, where $p+1 \leq r+2$, is given by

$$\begin{aligned}&\frac{(-1)^p}{k} \frac{r!\,(k+r-p)!}{p!\,(r-p)!\,(k-p-1)!} \\ &\qquad = \frac{(-1)^p}{k} \frac{r!\,(k+r-p)!\,(r-p+1)}{p!\,(r-p+1)!\,(k-p-1)!},\end{aligned} \tag{3}$$

the first formula failing to have a meaning when $p = r+1$. This formula follows from the usual proof by mathematical induction. There are $r+2$ terms in the brackets. If we put $p = r+1$ in (3), we get zero. Therefore, (2) is zero.

We now show how to calculate the terms of (1) not included in (2). The coefficients of $f_{r+1}(t), f_r(t), \cdots, f_2(t)$ can be found from (3) by replacing k by $r+1, r, \cdots, 2$ and p by $r, r-1, \cdots, 1$, respectively. The coefficient of $f_1(t)$ is $(r+1)!$ Since we do not need to know these coefficients explicitly, we do not compute them.

In order to find the coefficients of $f_n(t), f_{n+1}(t), \cdots, f_{n+r}(t)$, we consider the coefficient of $f_n(t)$. It is

$$(-1)^{r+1}\left[\frac{(n-1)!}{(n-r-1)!} - (r+1)\frac{n!}{(n-r)!} + \frac{(r+1)r}{2!}\frac{(n+1)!}{(n-r+1)!} - \cdots (-1)^r(r+1)\frac{(n+r-1)!}{(n-1)!}\right].$$

The sum of $p+1$ terms of this expression, where $p \leq r$, can be shown by mathematical induction to be

$$\frac{(-1)^p}{n}\frac{r!\,(n+p)!}{(r-p)!\,p!\,(n-r+p-1)!}, \qquad p \leq r. \tag{4}$$

There are $r+1$ terms in the bracket and the coefficient of $f_n(t)$ is given by (4) to be $(-1)^r\frac{(n+r)!}{n!}$.

The coefficients of $f_{n+1}(t), f_{n+2}(t), \cdots, f_{n+r}(t)$ can be found by replacing n by $n+1, n+2, n+3, \cdots, n+r$, and p by $r-1, r-2, \cdots, 0$, respectively, in formula (4).

Finally, we have the result:

$$\begin{aligned}\sum_{k=1}^{r-1}\frac{(k+r)!}{k!}\Delta^{r+1}f_k(t) &= c_1f_1(t) + c_2f_2(t) \\ &+ \cdots + c_{r+1}f_{r+1}(t) + (-1)^r\left[\frac{(n+r)!}{n!}f_n(t)\right. \\ &- r\frac{(n+r)!}{(n-1)!\,(n+1)}f_{n+1}(t) + \frac{r(r-1)}{2!}\frac{(n+r)!}{(n-1)!\,(n+2)}f_{n+2}(t) \\ &\left. - \cdots (-1)^r\frac{(n+r)!}{(n-1)!\,(n+r)}f_{n+r}(t)\right],\end{aligned} \tag{5}$$

where the c's are determined by (3) as explained.

The first $r+1$ terms are independent of n and bounded for all $t > 0$. Therefore, in order to show condition 1′ of Lemma 1 is satisfied, we have only to consider

$$\begin{aligned}&\frac{(n+r)!}{n!}f_n(t) - r\frac{(n+r)!}{(n-1)!\,(n+1)}f_{n+1}(t) \\ &+ \frac{r(r-1)}{2!}\frac{(n+r)!}{(n-1)!\,(n+2)}f_{n+2}(t) - \cdots (-1)^r\frac{(n+r)!}{(n-1)!\,(n+r)}f_{n+r}(t).\end{aligned} \tag{6}$$

We can write this in the form

$$\frac{(n+r)!}{(n-1)!}\left[\frac{f_n(t)}{n} - r\frac{f_{n+1}(t)}{n+1} + \frac{r(r-1)}{2!}\frac{f_{n+2}(t)}{n+2} - \cdots (-1)^r \frac{f_{n+r}(t)}{(n+r)}\right].$$

We denote $\frac{f_n(t)}{n}$ by $\varphi(n, t)$ and have

$$\frac{(n+r)!}{(n-1)!}\left[\varphi(n,t) - r\varphi(n+1,t) + \frac{r(r-1)}{2!}\varphi(n+2,t) - \cdots (-1)^r\varphi(n+r,t)\right] = \pm\frac{(n+r)!}{(n-1)!}[\varphi^{(r)}(n+\theta_r, t)],$$

where $0 < \theta_r < r$, by Lemma 3. We rewrite this as*

$$\frac{(n+r)!}{(n-1)!\,(n+\theta_r)^{r+1}}[(n+\theta_r)^{r+1}\varphi^{(r)}(n+\theta_r, t)].$$

The first factor is bounded for all n and all $t > 0$ as

$$\lim_{n\to\infty}\frac{(n+r)!}{(n-1)!\,(n+\theta_r)^{r+1}} = 1,$$

uniformly in t, since $0 < \theta_r < r$. Therefore, we need only consider the other factor. It can be written*

$$x^{r+1}\varphi^{(r)}(x,t), \quad \text{where} \quad \varphi(x,t) = \frac{f(x,t)}{x} \quad \text{and} \quad f(x,t) = e^{-tx\log x}.$$

From Lemma 2, we have

$$x^{r+1}\varphi^{(r)}(x,t) = \sum_{k=0}^{r}(-1)^k\frac{r!}{(r-k)!}x^{r-k}f^{(r-k)}(x,t)$$
$$= x^r f^{(r)}(x,t) - rx^{r-1}f^{(r-1)}(x,t) + \cdots (-1)^{r-1}r!\,f(x,t),$$

where $f^{(r)}(x, t)$ is given in sufficient detail by Lemma 4. We can surely write

$$(7) \quad |x^{r+1}\varphi^{(r)}(x,t)| \leq x^r|f^{(r)}(x,t)| + rx^{r-1}|f^{(r-1)}(x,t)| + \cdots + r!\,|f(x,t)|.$$

Call $tx\log x = \beta$. Then $tx(1+\log x) \leq 2tx\log x = 2\beta$. The first term of (7) gives, by Lemma 4,

$$|x^r f^{(r)}(x,t)| = e^{-\beta}\left[O(\beta^r) + \frac{1}{\log x}O(\beta^{r-1}) + \frac{1}{\log x}O(\beta^{r-2}) + \cdots + \frac{1}{\log x}O(\beta)\right] = e^{-\beta}\left[O(\beta^r) + \frac{1}{\log x}O(\beta^{r-1})\right].$$

For constant p, $\frac{\beta^p}{e^\beta}$ is bounded for all β (since it has a maximum for $\beta = p$

* The notation $\varphi^{(r)}(x, t)$ denotes differentiation with respect to the first argument.

and $\to 0$ as $\beta \to \infty$). It follows that

$$|x^r f^{(r)}(x, t)| \leq A + \frac{B}{\log x} < C,$$

where A, B, C are independent of x and t.

The other terms of (7), except the last, are exactly of the nature of the term investigated and consequently each of them is less than some constant, independent of x and t. The last term is $r!\, f(x, t) = r!\, e^{-tx \log x}$ which $< r!$ for all positive t and large positive x. Therefore, since the number of terms in (7) is finite, we can say $|x^{r+1}\varphi^{(r)}(x, t)| \leq M$, where M is independent of x and t. Therefore, expression (6) is bounded for all n and all $t > 0$, and condition 1 of Lemma 1 is satisfied.

It remains to show that condition 2 of Lemma 1 is satisfied. We desire to show that, for each constant $t > 0$, $\dfrac{(n+r)!}{n!}\Delta^{r+1}e^{-nt \log n}$ changes sign but a finite number of times and that this number is bounded independently of t. Since n is a positive integer, we need only consider

$$\Delta^{r+1}e^{-nt \log n}.$$

Lemma 3 states that

$$\Delta^r \varphi(n) = \varphi^{(r)}(n + \theta_r), \qquad 0 < \theta_r < r.$$

We therefore consider the $(r+1)$th derivative of $e^{-tx \log x}$.

Writing $\xi = 1 + \log x$,

$$f^{(r)}(x, t) = e^{-tx \log x} Q_r.$$

Since

$$f^{(r+1)}(x, t) = e^{-tx \log x}[-t\xi Q_r + Q_r'],$$

$$Q_{r+1} = Q_r' - t\xi Q_r. \tag{A}$$

We now say that

$$\begin{aligned} Q_r = &- \xi^{(r-1)}t + [r\xi\xi^{(r-2)} + \psi_{r-2}]t^2 \\ &- \left[\frac{r(r-1)}{2!}\xi^2\xi^{(r-3)} + \psi_{r-3}\right]t^3 + \cdots \\ &+ (-1)^{r-1}\left[\frac{r!}{(r-2)!\,2!}\xi^{r-2}\xi'\right]t^{r-1} + (-1)^r\xi^r t^r; \end{aligned} \tag{B}$$

where ψ_p is a polynomial in $\xi, \xi', \cdots, \xi^{(p-1)}$, in which the sum of the orders of the derivatives in each term is p, repetitions being counted, and in which the exponent of the highest power of ξ is less than $r - p - 1$, the exponent of the power of ξ in the first term of the same coefficient. This can be proved by mathematical induction, using (A).

We have, by differentiation,

$$Q_r' = -\xi^{(r)}t + [r\xi\xi^{(r-1)} + \psi_{r-1}]t^2 - \cdots + (-1)^p\left[\frac{r!}{(p-1)!\,(r-p+1)!}\xi^{p-1}\xi^{(r-p+1)} + \psi_{r-p+1}\right]t^p + \cdots + (-1)^r r\xi^{r-1}\xi' t^r.$$

We consider the expression for the coefficient of t^p in Q_r'. We recall that $\xi = 1 + \log x$, so that $\xi^{(p)} = (-1)^{p-1}\frac{(p-1)!}{x^p}$. The coefficient of t^p has a factor $\frac{1}{x^{r-p+1}}$. This is a factor of ψ_{r-p+1} because the sum of the orders of the derivatives for each term of polynomial ψ_{r-p+1} is $r - p + 1$. If we remove this factor, we have

$$\text{(C)} \qquad \frac{1}{x^{r-p+1}}\left[(-1)^r\frac{r!}{(r-p+1)!\,(p-1)!}\xi^{p-1} + P_{p-2}(\xi)\right]$$

where $P_{p-2}(\xi)$ is a polynomial in ξ of degree $\leq p-2$. This is true because ψ_{r-p+1} contains ξ to a power not higher than $p-2$.

ξ becomes positively infinite as $x \to \infty$. Therefore, we can select a value for x, say N_p, such that, for $x > N_p$, the sign of the expression (C) will be the sign of the first term; namely $(-1)^r$, as x, r, p, ξ are positive, $x > 1$. This sign is independent of p; thus, if for each p we select N_p so that the first term in each coefficient will dominate and then choose the largest of the set of values N_p, say N, we can say that for $x > N$ each coefficient of Q_r' will have the sign $(-1)^r$. It follows then that for $x > N$ and for all values of $t > 0$, Q_r' has one sign. This N is then independent of t. Therefore, Q_r can change sign not more than once, $x > N$. It follows then that $f^{(r)}(x, t) = f(x, t)Q_r$ changes sign not more than once, $x > N$. The location of this change of sign, if any, will generally depend on t, but N is independent of t.

Since this is true for any value of r, we have that $f^{(r+1)}(x, t)$ changes sign not more than once, $x > N$. We can then say that $\Delta^{r+1}e^{-nt\log n}$ can change sign not more than once for $n > N'$; where N' is the value of n corresponding to the value N for the $(r+1)$th derivative of $e^{-nt\log n}$. This N' can be taken independent of t also, since the two values differ by θ_r, where $0 < \theta_r < r$. Since we consider only integer values of n and since there are only a finite number of integers less than N', it follows that $\Delta^{r+1}e^{-nt\log n}$ can change sign but a finite number of times, bounded in t, for $t > 0$. Thus condition 2 of Lemma 1 is satisfied and the theorem is proved.

We can now state

THEOREM II. *The Dirichlet's series definition $D_{n \log n}$ includes C_r for all r, real or complex, $R(r) > -1$.*

It is known that if a series is summable C_r, it is summable C_s provided $R(s) > R(r) > -1$.* Theorem II follows immediately from this. When $R(r) \leq 0$, the regularity of the definition is sufficient to insure that it includes C_r.

3. THE TRANSFORMATION $D_{n \log n}$ FROM A DIRICHLET'S SERIES VIEWPOINT.

A Dirichlet's series† is a series of the form

$$f(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t},$$

where λ_n is a sequence of real increasing numbers whose limit is infinite; and in the cases we consider, $\lambda_1 \geq 0$. When t is complex, we write

$$t = \sigma + i\tau.$$

If we apply the transformation $D_{n \log n}$ to a series $\sum_{n=1}^{\infty} u_n$, we get

$$y(t) = \sum_{n=1}^{\infty} u_n e^{-nt \log n}$$

which is a Dirichlet's series, $\lambda_n = n \log n$. We now make use of this property of the transformation $D_{n \log n}$ to prove a theorem on regularity and to prove, in a much shorter way than in section 2, that $D_{n \log n}$ includes C_r.

THEOREM I. *The Dirichlet's series definition, $D_{n \log n}$, is regular if $t \to 0$ over a point set, T, which lies entirely within an angle whose vertex is at the origin such that $|\operatorname{am} t| \leq \mathbf{a} < \pi/2$.*

We assume $\sum_{n=1}^{\infty} u_n$ convergent and prove that $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges and $\lim_{\substack{t \to 0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-nt \log n} = \sum_{n=1}^{\infty} u_n$.

Since $\sum_{n=1}^{\infty} u_n$ converges, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges for $t = 0$ and to the value $\sum_{n=1}^{\infty} u_n$. Hardy and Riesz‡ prove that if a Dirichlet's series is convergent for $t = t_0$, then it is uniformly convergent throughout the angular

* Chapman, *Proceedings of the London Mathematical Society*, Series 2, Vol. 9 (1910–11), p. 369. See also *Report*, p. 27.

† Hardy and Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tracts in Mathematics and Mathematical Physics, No. 18).

‡ Loc. cit., Theorem 2.

region in the plane of t defined by the inequality

$$|\text{am } (t - t_0)| \leq \mathbf{a} < \pi/2.$$

Therefore, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges uniformly in this region, say to the value $f(t)$. It follows from this uniform convergence that if t approach 0 over any point set T in the region defined, $f(t)$ will approach $f(0)$ which is $\sum_{n=1}^{\infty} u_n$ and the theorem is proved.

Riesz* has given the following definition for the sum of a divergent series, $\sum_{n=1}^{\infty} u_n$. Form

$$y(t) = t^{-k} \sum_{\lambda_n < t} (t - \lambda_n)^k u_n$$

in which λ_n is a sequence of real increasing numbers whose limit is infinite, $\lambda_1 \geq 0$. Then the generalized value of the series is given by

$$\lim_{\substack{t \to \infty \\ T}} y(t)$$

where T is a set of positive real numbers. If this limit exists, the series is said to be summable (λ, k). This definition is regular, $k \geq 0$.

Riesz has shown† that, if $\lambda_n = n$, then this definition is equivalent to Cesàro's means and to the generalizations of the Cesàro means due to Knopp and Chapman.

LEMMA: *If* $\sum_{n=1}^{\infty} u_n$ *is summable* (n, k), *it is summable* $(n \log n, k)$ *to the same value.*

This can be shown by use of a theorem due to Hardy:‡

If the series $\sum_{n=1}^{\infty} u_n$ is summable (λ, k) to the sum l, and if μ is a logarithmico-exponential function of λ such that $\mu = O(\lambda^{\Delta})$, where Δ is a constant; then the series, $\sum_{n=1}^{\infty} u_n$, is summable (μ, k) to the sum l.

In this case, let $\mu = n \log n$ and $\lambda = n$; then the condition $n \log n = O(n^{\Delta})$ is true for any constant $\Delta > 1$. The hypotheses of the theorem are satisfied and the lemma is proved.

We are now in a position to prove

THEOREM II. *The Dirichlet's series definition,* $D_{n \log n}$, *when* t *approaches* 0 *over a point set* T, *lying within an angle with vertex at the origin such that* $|\text{am } t| \leq \mathbf{a} < \pi/2$, *includes the Cesàro means of all orders.*

* *Comptes Rendus*, Vol. 149 (1909), p. 910.

† *Comptes Rendus*, Vol. 152 (1911), p. 1651.

‡ *Proceedings of London Mathematical Society*, Series 2, Vol. 15 (1916), p. 72.

To prove this theorem, it is necessary to show that if $\sum_{n=1}^{\infty} u_n$ is summable C_k to the sum l, then $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges, $R(t) > 0$, and $\lim_{t \to 0} \sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is l.

We assume that $\sum_{n=1}^{\infty} u_n$ is summable C_k and, consequently, summable (n, k) to the sum l, where k is the order of summability. It is evident that, for $t = 0$, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is summable (n, k) and, by the lemma, $(n \log n, k)$ and that the sum is l. It is also summable $(n \log n, k)$ for all t whose real part is greater than 0, say to the sum $f(t)$. This is due to a theorem by Hardy and Riesz* which states that if a Dirichlet's series, $\sum_{n=1}^{\infty} u_n e^{-\lambda_n t}$, is summable (λ, k) for a value of t whose real part is σ, then it is summable (λ, k) for all values of t whose real part is greater than σ.

Hardy and Riesz† also proved that if a series is summable (λ, k) for $t = t_0$, and has sum $f(t_0)$, and if the series has sum $f(t)$ when the real part of t is greater than the real part of t_0, then $f(t) \to f(t_0)$ as $t \to t_0$ along any path lying entirely within the angle whose vertex is at t_0 and such that $|\text{am } (t - t_0)| \leq \mathbf{a} < \pi/2$. We have then that, if t approaches zero along any path in the angle defined, $f(t) \to l$.

But, we shall show that $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is actually convergent, $R(t) > 0$. Since $\sum_{n=1}^{\infty} u^n$ is summable C_k, $u_n/n^k \to 0$ as $n \to \infty$. The series $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ has a region of absolute convergence coinciding with its region of convergence.‡ Therefore, we can write

$$|u_n e^{-nt \log n}|^{1/n} = \left|\frac{u_n}{n^k}\right|^{1/n} \left|\frac{n^{k/n}}{n^t}\right| = \left|\frac{u_n}{n^k}\right|^{1/n} |n^{(k/n)-t}| \to 0 \quad \text{for} \quad R(t) > 0.$$

Therefore, by Cauchy's test $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is convergent, $R(t) > 0$.

Because of the regularity of $(n \log n, k)$, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ must be summable $(n \log n, k)$ to the same $f(t)$ to which it converges. It follows then that the value approached as $t \to 0$ must be the same in both cases. There-

* Loc. cit., Theorem 25.

† Loc. cit., Theorem 28.

‡ Hardy and Riesz, loc. cit., Theorem 9.

fore,

$$\lim_{t\to 0}\sum_{n=1}^{\infty} u_n e^{-nt\log n} = l$$

and the theorem is proved.

4. SOME RELATIONS BETWEEN THE BOREL DEFINITION, THE LEROY DEFINITION, AND THE DIRICHLET'S SERIES DEFINITION $D_{n\log n}$.

THEOREM I. *LeRoy's definition does not include Borel's definition.*

We prove this theorem by giving a series that is summable Borel and not summable LeRoy. Let us consider the series

$$\sum_{n=0}^{\infty} u_n = 1 + 0 - 2 + 0 + 3\cdot 4 + 0 - 4\cdot 5\cdot 6 + 0 + \cdots,$$

where

$$\begin{cases} u_n = 0, & n \text{ odd}, \\ u_n = (-1)^{n/2}\dfrac{n!}{\left(\dfrac{n}{2}\right)!}, & n \text{ even}, \end{cases}$$

This series is summable Borel, for

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{u_n t^n}{n!} &= 1 + 0 - \frac{2t^2}{2!} + 0 + \frac{3\cdot 4t^4}{4!} + 0\cdots \\ &= 1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \cdots \\ &= e^{-t^2}, \end{aligned}$$

and

$$\int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}\frac{u_n t^n}{n!}\,dt = \int_0^{\infty} e^{-t^2-t}dt$$

which converges, since $e^{-t^2-t} \le e^{-t}$, both are positive, and $\int_0^{\infty} e^{-t}dt$ converges. Therefore, the series is summable Borel.

This series is not summable LeRoy. If we apply the LeRoy transformation to this series, we get

$$\begin{aligned} L = 1 + 0 - \frac{\Gamma(2t+1)}{2!}2 + 0 + \frac{\Gamma(4t+1)}{4!}3\cdot 4 + 0 \\ - \frac{\Gamma(6t+1)}{6!}4\cdot 5\cdot 6 + 0 + \frac{\Gamma(8t+1)}{8!}5\cdot 6\cdot 7\cdot 8\cdots \end{aligned}$$

so that, if V_n is the nth term of this series,

$$\begin{cases} V_n = 0, & n \text{ odd}, \\ V_n = (-1)^{n/2}\dfrac{\Gamma(nt+1)}{\left(\dfrac{n}{2}\right)!}, & n \text{ even}. \end{cases}$$

In order to see what happens to the nth term of this series, as $n \to \infty$, we write $n = 2k$, so that

$$V_{2k} = (-1)^k \frac{\Gamma(2kt+1)}{k!}$$

and find the limit approached by V_{2k} as $k \to \infty$. By Stirling's formula

$$\frac{\Gamma(2kt+1)}{k!} = 2^{2kt+(1/2)} k^{k(2t-1)} t^{2kt+(1/2)} e^{-k(2t-1)}[1 + o(1)],$$

so that

$$\left[\frac{\Gamma(2kt+1)}{k!}\right]^{1/2k} = 2^{t+(1/4k)} k^{t-(1/2)} t^{t+(1/4k)} e^{-t+(1/2)}[1 + o(1)].$$

For a fixed $t > \frac{1}{2}$,

$$\left[\frac{\Gamma(2kt+1)}{k!}\right]^{1/2k} \to \infty \text{ as } k \to \infty.$$

It follows then that the series L cannot converge, $t > \frac{1}{2}$, since lim sup $|V_n| = \infty$. Since the series L does not converge, $t_0 < t < 1$, it is not summable LeRoy and the theorem is proved.

Before proceeding to the next theorem, we prove two lemmas.

LEMMA 1. *If* $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *converges, then* $\sum_{n=0}^{\infty} \frac{\Gamma(nt'+1)}{\Gamma(n+1)} u_n$ *converges absolutely,* $t' < t$.

We assume that $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ is convergent. Therefore,

$$\frac{\Gamma(nt+1)}{\Gamma(n+1)} |u_n| < C, \quad |u_n| < C \frac{\Gamma(n+1)}{\Gamma(nt+1)},$$

and

$$\frac{\Gamma(nt'+1)}{\Gamma(n+1)} |u_n| \leq \frac{\Gamma(nt'+1)}{\Gamma(n+1)} \cdot C \frac{\Gamma(n+1)}{\Gamma(nt+1)} = \frac{\Gamma(nt'+1)}{\Gamma(nt+1)} C$$

$$= n^{n(t'-t)} e^{-n(t'-t)} \frac{t'^{nt'+(1/2)}}{t^{nt+(1/2)}} C[1 + o(1)],$$

then

$$\left|\frac{\Gamma(nt'+1)}{\Gamma(n+1)} u_n\right|^{1/n} \leq n^{t'-t} e^{-(t'-t)} \frac{t'^{t'+(1/2n)}}{t^{t+(1/2n)}} C^{1/n}[1 + o(1)]$$

$$< kn^{t'-t}[1 + o(1)] \to 0, \text{ as } n \to \infty, \ t' < t.$$

Thus Lemma 1 is proved.

LEMMA 2. *Let* $k(a, x)$ *be continuous,* $a > 0$, $x > 0$, *and let* $U(x)$ *be a continuous function such that* $\lim_{x \to \infty} U(x) = l$. *If*

1. $\int_0^{\infty} |k(a, x)| dx$ *exists for each* $a > 0$ *and is bounded for all* $a > 0$,

2. $\lim_{a\to 0}\int_{\alpha}^{\beta} |k(a, x)|\,dx = 0,\ \beta > \alpha > 0,$

3. $\lim_{a\to 0}\int_{0}^{\alpha} k(a, x)dx = -1,\ \alpha > 0,$

4. $\lim_{a\to 0}\int_{0}^{\alpha} k(a, x)dx = 0,$

then $\int_0^\infty k(a, x)U(x)dx$ *exists for each* $a > 0$ *and*

$$\lim_{a\to 0}\int_0^\infty k(a, x)U(x)dx = l - U(0).$$

The fact that $\int_0^\infty k(a, x)U(x)dx$ exists follows immediately from condition 1 and the fact that $U(x)$ is bounded.

We can write

$$\int_0^\infty k(a, x)U(x)dx = \int_0^\eta k(a, x)[U(x) - U(0)]dx + U(0)\int_0^\eta k(a, x)dx + \int_\eta^\xi k(a, x)U(x)dx + \int_\xi^\infty k(a, x)[U(x) - l]dx + \int_\xi^\infty k(a, x)dx,$$

where ξ, η are constants, each independent of a, such that

$$\begin{aligned} |U(x) - l| &< \epsilon, \quad x > \xi \\ |U(x) - U(0)| &< \epsilon, \quad x < \eta. \end{aligned}$$

Because of (3)

$$U(0)\int_0^\eta k(a, x)dx = -U(0) + h_1(a), \qquad \text{where} \qquad \lim_{a\to 0} h_1(a) = 0,$$

and because of (3) and (4)

$$l\int_\xi^\infty k(a, x)dx = l + h_2(a), \qquad \text{where} \qquad \lim_{a\to 0} h_2(a) = 0.$$

It follows that

$$\left|\int_0^\infty k(a, x)U(x)dx + U(0) - l\right| \le \int_0^\eta |k(a, x)|\,|U(x) - U(0)|\,dx + \int_\eta^\xi |k(a, x)|\,|U(x)|\,dx + \int_\xi^\infty |k(a, x)|\,|U(x) - l|\,dx.$$

By (1) and the choice of η

$$\int_0^\eta |k(a, x)|\,|U(x) - U(0)|\,dx \le \epsilon \cdot C;$$

by (2) and the fact that $|U(x)| < M$

$$\int_\eta^\xi |k(a, x)|\,|U(x)|\,dx \leq M \cdot h_3(a), \qquad \text{where} \qquad \lim_{a \to 0} h_3(a) = 0;$$

by (1) and the choice of ξ

$$\int_\xi^\infty |k(a, x)|\,|U(x) - l|\,dx \leq \epsilon \cdot A.$$

It follows that

$$\left| \int_0^\infty k(a, x)U(x)dx + U(0) - l \right| \leq \epsilon \cdot C + M \cdot h_3(a) + \epsilon \cdot A + |h_1(a)| + |h_2(a)|,$$

so that

$$\limsup_{a \to 0} \left| \int_0^\infty k(a, x)U(x)dx + U(0) - l \right| \leq \epsilon \cdot C + \epsilon \cdot A.$$

Since this limit is less than an arbitrarily chosen small positive constant, it follows that

$$\lim_{a \to 0} \int_0^\infty k(a, x)U(x)dx = l - U(0).$$

THEOREM II. *If a series,* $\sum_{n=0}^\infty u_n$, *is summable Borel and if* $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *converges,* $t_0 < t < 1$, *then the series is summable LeRoy, and to the same value.*

Bromwich proved* this theorem under the assumption that

$$\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$$

converges absolutely. Because of Lemma 1, we need not require the absolute convergence of this series but simply its convergence. The following proof differs materially from the one given by Bromwich.

We assume that $\int_0^\infty e^{-x}u(x)dx$ converges to the value l, and desire to prove that, if $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ converges, then $\lim_{t \to 1^-} \sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = l$. We can express $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ in terms of an integral as follows:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, \qquad \frac{\Gamma(nt+1)}{n!} = \int_0^\infty e^{-x}\frac{x^{nt}}{n!}\,dx,$$

and

$$\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = \sum_{n=0}^\infty \int_0^\infty e^{-x}\frac{u_n x^{nt}}{n!}\,dx = \int_0^\infty e^{-x} \sum_{n=0}^\infty \frac{u_n x^{nt}}{n!}\,dx,$$

* Bromwich, *Infinite Series* (London, 1908).

where the term-by-term integration is justified by Bromwich* under the hypothesis of absolute convergence which is equivalent, because of Lemma 1, to ours.

We can now write

$$\lim_{t\to 1^-}\sum_{n=0}^{\infty}\frac{\Gamma(nt+1)}{\Gamma(n+1)}u_n=\lim_{t\to 1^-}\int_0^{\infty}e^{-x}u(x^t)dx,\quad\text{where}\quad u(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}u_n.$$

Now, if the change of variable $x=y^{1/t}$ is made, and if we then replace y by x and $1/t-1$ by a, we have

$$\lim_{t\to 1^-}\int_0^{\infty}e^{-x}u(x^t)dx=\lim_{a\to 0^+}\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx+\lim_{a\to 0^+}a\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx.$$

It now remains to show that if $\int_0^{\infty}e^{-x}u(x)dx$ converges to the value l, then $\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx$ converges for all values of a in an interval $(0, a)$ and that $\lim_{a\to 0}\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx=l$. If this is proved, it will be clear that the second integral in the above equation $\to 0$ as $a\to 0$.

Let us write $U(x)=\int_0^x e^{-s}u(s)ds$. Then it follows that $\lim_{n\to\infty}U(x)=l$ and that $U'(x)=e^{-x}u(x)$. Therefore,

$$\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx=\int_0^{\infty}e^{x-x^{1+a}}x^a U'(x)dx.$$

Integration by parts gives

$$\int_0^{\infty}e^{x-x^{1+a}}x^a U'(x)dx=\lim_{\xi\to\infty}\left[e^{x-x^{1+a}}x^a U(x)\right]_0^{\xi}-\lim_{\xi\to\infty}\int_0^{\infty}\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx.$$

The last integral converges as $\xi\to\infty$ and we have

$$\int_0^{\infty}e^{-x^{1+a}}x^a u(x)dx=-\int_0^{\infty}\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx.$$

Lemma 2 can now be used to show that whenever $\lim_{x\to\infty}U(x)=l$, then

$$\int_0^{\infty}-\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx$$

exists for each $a>0$ and

$$\lim_{a\to 0}\int_0^{\infty}-\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx=l.$$

Here

$$k(a,x)=-\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]$$

is continuous, $a > 0$, $x > 0$. $U(x)$ is an indefinite integral and therefore continuous, $0 < x$, and $\lim_{x \to \infty} U(x) = l$. It remains to show that $k(a, x)$ satisfies the four conditions of Lemma 2.

In order to show that condition 1 is satisfied, it is desirable to know where $k(a, x)$ changes sign. We find

$$k(a, x) = -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right] = -e^{x-x^{1+a}}x^{a-1}\left[x + a - (a+1)x^{a+1}\right].$$

Since the factor outside the bracket is negative, we investigate $x + a - (1 + a)x^{1+a}$. It vanishes when $x = 1$; and its second derivative < 0, $x > 0$. Since the function > 0 when $x = 0$, it follows that it changes sign only when $x = 1$. Therefore, $k(a, x)$ is negative, $0 < x < 1$, and positive, $x > 1$. We now show that the four conditions of Lemma 2 are satisfied.
Condition 1.

$$\int_0^\infty \left| -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right] \right| dx = \int_0^1 \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx - \int_1^\infty \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx$$
$$= \left[e^{x-x^{1+a}}x^a\right]_0^1 - \left[e^{x-x^{1+a}}x^a\right]_1^\infty = 2.$$

Condition 2.

If $\alpha < \beta \leq 1$ or $1 \leq \alpha < \beta$, then

$$\lim_{a \to 0} \int_\alpha^\beta \left| -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right] \right| dx = \lim_{a \to 0} \int_\alpha^\beta \pm \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx$$
$$= \pm \lim_{a \to 0}\left[e^{\beta-\beta^{1+a}}\beta^a - e^{\alpha-\alpha^{1+a}}\alpha^a\right] = 0.$$

If $\alpha < 1 < \beta$, then

$$\lim_{a \to 0} \int_\alpha^\beta \left| -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right] \right| dx = \lim_{a \to 0} \int_\alpha^1 \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx$$
$$- \lim_{a \to 0} \int_1^\beta \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx = \lim_{a \to 0}\left[e^{x-x^{1+a}}x^a\right]_\alpha^1 - \lim_{a \to 0}\left[e^{x-x^{1+a}}x^a\right]_1^\beta = 0.$$

Condition 3.

If $\alpha > 0$,

$$\lim_{a \to 0} \int_0^\alpha -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx = \lim_{a \to 0}\left[-e^{x-x^{1+a}}x^a\right]_0^\alpha = -1.$$

Condition 4.

$$\int_0^\infty -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]dx = \left[-e^{x-x^{1+a}}x^a\right]_0^\infty = 0.$$

The conditions of the lemma are then satisfied and we can say that if $\int_0^\infty e^{-x}u(x)dx$ converges to the value l, then

$$\int_0^\infty e^{-x^{1+a}}x^a u(x)dx$$

converges, $a > 0$, and

$$\lim_{a \to 0} \int_0^\infty e^{-x^{1+a}} x^a u(x) dx = l - U(0).$$

But $U(0) = 0$, and the theorem is proved.

COROLLARY. *The LeRoy and Borel definitions are mutually consistent.**

THEOREM III. *A sufficient condition that* $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *shall converge,* $t_0 < t < 1$, is that $\lim_{n \to \infty} \frac{|u_n|^{1/n}}{n^\epsilon} = 0$, $\epsilon > 0$.

By Stirling's formula

$$\frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = n^{n(t-1)} t^{nt+(1/2)} e^{-n(t-1)} [1 + o(1)] u_n.$$

Therefore,

$$\left| \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n \right|^{1/n} = n^{t-1} t^{t+(1/2n)} e^{-(t-1)} [1 + o(1)] |u_n|^{1/n} \to 0$$

under our hypothesis, $t_0 < t < 1$.

THEOREM IV. *If a given series,* $\sum_{n=0}^{\infty} u_n$, *is summable Borel and if* $\lim_{n \to \infty} \frac{|u_n|^{1/n}}{n^\epsilon} = 0$, $\epsilon > 0$, *then the series is summable LeRoy to the same value.*

The proof follows immediately from Theorem II and Theorem III.

THEOREM V. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* C_r *and summable Borel,*† *it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable C_r, it is necessary that $\frac{u_n}{n^r} \to 0$ as $n \to \infty$. Therefore,

$$|u_n| < Cn^r, \qquad |u_n|^{1/n} < C^{1/n} n^{r/n}, \qquad \text{and}$$

$$\frac{|u_n|^{1/n}}{n^\epsilon} \leq \frac{C^{1/n} n^{r/n}}{n^\epsilon} \leq C^{1/n} n^{(r/n)-\epsilon} \to 0, \quad n \to \infty,$$

and, by Theorem IV, the theorem is proved.

THEOREM VI. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* E_r§ *and Borel,*‡ *it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable E_r, it is necessary that $\frac{u_n}{(2r-1)^n} \to 0$ as $n \to \infty$.*

* Ricotti has proved this corollary in another way. See *Giornale di Matematiche*, Vol. 48 (1910), p. 80.

† Necessarily to the same value because of mutual consistency.

‡ *Report*, p. 32.

§ Necessarily to the same value because of mutual consistency.

Therefore,

$$|u_n|^{1/n} < C^{1/n}(2r - 1),$$

and

$$\frac{|u_n|^{1/n}}{n^\epsilon} \leq \frac{C^{1/n}(2r-1)}{n^\epsilon} \to 0, \quad \text{as} \quad n \to \infty,$$

and, by Theorem IV, the theorem is proved.

THEOREM VII. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* E_r, *it is summable LeRoy to the same value.*

This theorem states that LeRoy's definition includes the definition given by the exponential mean, E_r. Borel's definition includes E_r.* Therefore, every series summable E_r is summable Borel and consequently, by Theorem VI, is summable LeRoy.

THEOREM VIII. *If* $\sum_{n=0}^{\infty} u_n$ *is summable by the Euler power series definition*† *and summable Borel,*‡ *it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable by the Euler power series definition, it is necessary that $\limsup_{n\to\infty} |u_n|^{1/n} \leq 1$.§ Therefore,

$$\lim_{n\to\infty} \frac{|u_n|^{1/n}}{n^\epsilon} \leq \lim_{n\to\infty} \frac{1}{n^\epsilon} = 0, \quad \epsilon > 0,$$

and the theorem is proved.

THEOREM IX. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* $D_{n \log n}$ *and Borel, then it is summable LeRoy to the value to which it is summable Borel.*

Given an $\epsilon > 0$, then it is necessary if the $D_{n \log n}$ series converges, $t > 0$, that

$$|u_n| e^{-nt \log n} < A, \quad t > 0,$$

and, in particular,

$$|u_n| e^{-n(\epsilon/2) \log n} < A.$$

Then

$$|u_n| < A e^{n(\epsilon/2) \log n}$$

and

$$|u_n|^{1/n} < A^{1/n} e^{\epsilon/2 \log n} = A^{1/n} n^{\epsilon/2}.$$

* *Report*, p. 27.
† *Report*, p. 23.
‡ Necessarily to the same value because of mutual consistency.
§ *Report*, p. 32.

Therefore,

$$\frac{|u_n|^{1/n}}{n^\epsilon} < \frac{A^{1/n}n^{\epsilon/2}}{n^\epsilon} = \frac{A^{1/n}}{n^{\epsilon/2}} \to 0,$$

and, by Theorem IV, the theorem is proved.

THEOREM X. *The Dirichlet's series definition $D_{n \log n}$ does not include Borel's definition.*

This is shown by means of the series

$$\sum_{n=1}^{\infty} u_n = 1 + 0 - 2 + 0 + 3\cdot 4 + 0 - 4\cdot 5\cdot 6 + \cdots,$$

where

$$\begin{cases} u_n = 0, & n \text{ even}, \\ u_n = (-1)^{(n-1)/2} \dfrac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!}, & n \text{ odd}. \end{cases}$$

It was shown in the proof of Theorem I of this section that this series is Borel summable. It is not summable $D_{n \log n}$. To show this, we form

$$\sum_{n=1}^{\infty} u_n e^{-nt \log n}$$

and show that it is not convergent $t > 0$. The nth term is

$$\begin{cases} 0, & n \text{ even}, \\ (-1)^{(n-1)/2} \dfrac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!} e^{-nt \log n}; & n \text{ odd}. \end{cases}$$

We take the superior limit of the nth root of the nth term as $n \to \infty$.

$$\limsup_{n\to\infty} \left| \frac{(n-1)!}{\left(\frac{n-1}{2}\right)!} e^{-nt \log n} \right|^{1/n} = \begin{cases} 0, & t \geq \frac{1}{2}, \\ \infty, & t < \frac{1}{2}. \end{cases}$$

By Stirling's formula

$$(n-1)! = \Gamma(n) = \sqrt{2\pi}\, n^{n-(1/2)} e^{-n}[1 + o(1)],$$

and

$$\frac{n-1}{2}! = \Gamma\left(\frac{n+1}{2}\right) = \sqrt{2\pi}\left(\frac{n+1}{2}\right)^{(n+1)/2-1/2} e^{-(n+1)/2}[1 + o(1)],$$

and

$$\left| \frac{(n-1)!}{\left(\frac{n-1}{2}\right)!} e^{-nt \log n} \right|^{1/n} = \sqrt{2}\, n^{1/2-1/2n} \left(1 + \frac{1}{n}\right)^{-1/2} e^{-1/2+1/2n} n^{-t}[1 + o(1)].$$

$$= \sqrt{2}\, n^{1/2-t-1/2n}\left(1+\frac{1}{n}\right)^{-1/2} e^{-1/2+1/2n}[1+o(1)]$$

$$= \begin{cases} 0, & t \geq \frac{1}{2} \\ \infty, & t < \frac{1}{2}. \end{cases}$$

Since for this Dirichlet's series the region of convergence coincides with the region of absolute convergence,* it follows that this series does not converge, $0 < t < \frac{1}{2}$, and the theorem is proved.

THEOREM XI. *Borel's definition does not include* $D_{n \log n}$.

We have proved that $D_{n \log n}$ includes C_r. A series summable C_r and not Borel would be summable $D_{n \log n}$ and not Borel. Since the Borel definition does not include C_r, such series exist.

5. SOME INDICATIONS OF THE RELATION BETWEEN THE LEROY DEFINITION AND C_r.

We have attempted, by the method used in section 2 for the proof that $D_{n \log n}$ includes C_r, to show that the LeRoy definition includes C_r. To do this, it would be sufficient to prove

$$1.\ \lim_{t \to 1} \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 1,$$

$$2.\ \lim_{n \to \infty} n^r \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 0, \quad t < 1,$$

$$3.\ \sum_{n=0}^{\infty} n^r \left| \Delta^{r+1} \frac{\Gamma(nt+1)}{\Gamma(n+1)} \right|$$

converges for each $t < 1$ and is bounded for all $t < 1$.

It is easily seen that the first condition is satisfied. By using Stirling's formula, condition 2 can be shown to be satisfied for all values of r.

In order to show that condition 3 is satisfied, it would be sufficient to show that conditions 1′ and 2 of Lemma 1, section 2, are satisfied.

For the case $r = 1$, we shall prove that condition 1′ is satisfied. To show this, it is necessary to prove that

$$\sum_{k=0}^{n} k\Delta^2 \frac{\Gamma(kt+1)}{\Gamma(k+1)}$$

is bounded for all n and all t, $0 < t < 1$.

If we write $f_k(t)$ for $\dfrac{\Gamma(kt+1)}{\Gamma(k+1)}$, we have

$$S_n(t) = \sum_{k=0}^{n-1} k\Delta^2 f_k(t) = f_1(t) - nf_n(t) + (n-1)f_{n+1}(t);$$

analogous to formula (5), section 2. The first term of $S_n(t)$ is independent

* Hardy and Riesz, loc. cit., Theorem 9, page 9.

of n and bounded for all t under consideration. Therefore, we need consider only the last two terms.

Replacing $f_n(t)$ by $\dfrac{\Gamma(nt+1)}{\Gamma(n+1)}$, the last two terms of $S_n(t)$ give

$$S_n'(t) = (n-1)\frac{\Gamma(\overline{n+1}\,t+1)}{\Gamma(n+2)} - \frac{n\Gamma(nt+1)}{\Gamma(n+1)}$$

$$= \frac{(n-1)t\Gamma(\overline{n+1}t)}{n\Gamma(n)} - \frac{nt\Gamma(nt)}{\Gamma(n)}.$$

Since both terms of $S_n'(t)$ are bounded for $n \leq n_0$, $t \leq t_0 < 1$, it is necessary only to consider what happens to $S_n'(t)$ when n is large and t near 1.

We can write

$$S_n'(t) = \frac{\left(1-\frac{1}{n}\right)t\Gamma(\overline{n+1}t) - nt\Gamma(nt)}{\Gamma(n)}.$$

Applying Stirling's formula to the gamma functions, we have

$$S_n'(t) = n^{-n\delta+1}t^{nt+(1/2)}e^{n\delta+O(1/n)}$$

$$\times\left[\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{nt+t-(1/2)}n^{-\delta}t^t e^{-t+O(1/n)} - 1\right],$$

where $\delta = 1 - t$.

If we write each term as an exponential and expand the logarithms into series, we get

$$(1)\quad t^{nt+(1/2)}e^{n\delta+O(1/n)} = e^{-(1/2)\delta+O(n\delta^2)+O(1/n)} = e^{O(n\delta^2)}\left[1+O(\delta)+O\left(\frac{1}{n}\right)\right],$$

$$(2)\quad \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{nt+t-(1/2)}e^{-t+O(1/n)} = \left[1+O\left(\frac{1}{n}\right)\right],$$

and

$$(3)\qquad t^t = 1 + O(\delta).$$

By the law of the mean and the fact that $\delta > 0$, we have

$$0 < 1 - n^{-\delta} < \delta \log n,$$

so that

$$(4)\qquad n^{-\delta} = 1 + O(\delta \log n).$$

Substituting (1), (2), (3), (4), in $S_n'(t)$, we get

$$S_n'(t) = n^{-n\delta+1}e^{O(n\delta^2)}\left[O\left(\frac{1}{n}\right) + O(\delta \log n)\right]$$

$$= e^{-n\delta \log n + O(n\delta^2)}[O(1) + O(n\delta \log n)].$$

But

$$O(n\delta^2) = [n\delta \log n]O\left(\frac{\delta}{\log n}\right) < \tfrac{1}{2}n\delta \log n$$

for n sufficiently large and t near 1. Therefore,

$$e^{O(n\delta^2)} < e^{(1/2)n\delta \log n}$$

and

$$\begin{aligned} S_n'(t) &= e^{-(1/2)n\delta \log n}[O(1) + O(n\delta \log n)] \\ &= O(e^{-(1/2)n\delta \log n}) + O(n\delta e^{-(1/2)n\delta \log n} \log n). \end{aligned}$$

Let $y = \frac{1}{2}n\delta \log n$, then

$$S_n'(t) = O(e^{-y}) + O(ye^{-y}).$$

Since y is positive, $S_n'(t)$ is bounded for all n and all $t < 1$; and condition $1'$ of the lemma is satisfied.

It seems not unlikely that this proof can be carried through for higher values of r. However, in order to complete the proof, even for the case $r = 1$, it would be necessary to show that condition 2 of the lemma is satisfied. We have not been able to do this. We are unable to determine accurately the number of changes of sign for $\Delta^2 \dfrac{\Gamma(nt+1)}{\Gamma(n+1)}$. We succeeded, however, by drawing graphs for $\dfrac{\Gamma(tx)}{\Gamma(x)}$ for various values of t near 1, in making it appear very likely that $\dfrac{d^2}{dx^2}\left[\dfrac{\Gamma(tx)}{\Gamma(x)}\right]$ changes sign just once for t near 1.* While this does not constitute a proof, there seems but little doubt that LeRoy's definition includes C_1, and probably C_r. This belief is somewhat strengthened by the fact that Hardy's definition,

$$\lim_{t \to 0} \sum_{n=0}^{\infty} e^{-nt \log n} u_n,$$

which he considered "substantially equivalent" † to LeRoy's, includes C_r.‡

* For $t = .9$, the second derivative appeared to change sign for a value of x near 3.

† See Introduction.

‡ Our proof that $D_{n \log n}$ includes C_r can be carried through with but minor changes for Hardy's definition.

MODULAR INVARIANTS OF A BINARY GROUP WITH COMPOSITE MODULUS.

By Constance R. Ballantine.

1. Let $\mathfrak{G}$ be the group of transformations

$$\mathfrak{T}: \begin{cases} x' = ax + by, \\ y' = cx + dy, \end{cases}$$

also denoted by

$$\mathfrak{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

on the indeterminates x and y, where a, b, c, d are integers, and

$$\Delta = ad - bc$$

is prime to a given integer m. A rational integral function, $F(x, y)$, with integral coefficients, will be called an invariant of $\mathfrak{G}$ modulo m if

$$F(x', y') \equiv \Delta^{\lambda} F(x, y) \pmod{m},$$

identically in x, y, a, b, c, d, after x', y', Δ have been replaced by their expressions above.

We shall frequently find it convenient to work, not with the infinite group $\mathfrak{G}$, but with the finite group, denoted by $\mathfrak{G}'$, of *classes* $\mathfrak{T}'$ of all transformations congruent to a given $\mathfrak{T}$ modulo m. We shall write

$$\mathfrak{T}_1 \equiv \mathfrak{T} \pmod{m},$$

if

$$a_1 \equiv a, \quad b_1 \equiv b, \quad c_1 \equiv c, \quad d_1 \equiv d \pmod{m}.$$

Thus, if $\mathfrak{T}_1$ belongs to $\mathfrak{T}'$,

$$F(a_1x + b_1y,\ c_1x + d_1y) \equiv F(ax + by,\ cx + dy) \pmod{m},$$

identically in x and y. Hence we may state

Theorem I. *The invariance of a rational integral function with integral coefficients under the finite group $\mathfrak{G}'$ modulo m is equivalent to its invariance under $\mathfrak{G}$ modulo m.*

2. Professor L. E. Dickson* has treated the case $m = p$, a prime. Restricting attention at first to the group H (with corresponding finite group H') for which

$$\Delta \equiv 1 \pmod{p},$$

he has shown later that the fundamental system of invariants is the same

* *Madison Colloquium Lectures*, pp. 33 ff.

as when Δ is merely prime to p. This fundamental system he has found to be

$$L = x^p y - xy^p,$$
$$Q = \frac{x^{p^2} y - xy^{p^2}}{L}.$$

The group H consists of all transformations

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1 \pmod{p},$$

where a, b, c, d are integers; and the finite group H' is the group of classes S' of all transformations

$$S \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{p},$$

where $\alpha, \beta, \gamma, \delta$ are integers reduced modulo p.

3. Passing now to the case of a composite modulus

$$\pi = p_1 p_2 \cdots p_n,$$

where the p_i are distinct primes, we wish to find all rational integral functions $I(x, y)$ with integral coefficients which are invariant modulo π under the group Γ of transformations

$$\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1 \pmod{\pi},$$

where a, b, c, d are integers; i.e., all functions $I(x, y)$ for which

$$I(ax + by, cx + dy) \equiv I(x, y) \pmod{\pi},$$

identically in x and y, for all admissible a, b, c, d. By Theorem I, this is equivalent to finding all the invariants, modulo π, of the group Γ' of classes Θ' of transformations

$$\Theta \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{\pi},$$

where $\alpha, \beta, \gamma, \delta$ are integers reduced modulo π.

4. Let p denote any one of the p_i, and m the product of the others, so that $\pi = pm$. Consider the sub-group G' of Γ', composed of all classes of transformations of Γ which are congruent to identity modulo m. These transformations, taken altogether, form the corresponding infinite sub-group G, but it is with the finite sub-group that we shall chiefly be dealing. Every class of transformations of G' is composed of transformations

$$T \equiv \begin{pmatrix} \alpha & m\beta \\ m\gamma & \delta \end{pmatrix} \pmod{\pi}, \qquad \alpha \equiv \delta \equiv 1 \pmod{m},$$

where α, $m\beta$, $m\gamma$, δ are integers reduced modulo π; while, since T belongs to Γ, we have

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{\pi}.$$

The latter congruence, being satisfied identically modulo m, requires merely that

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{p}.$$

Since α and δ are residues of integers modulo π and are congruent to unity modulo m, they range over the values

$$1,\ m+1,\ 2m+1,\ \cdots,\ (p-1)m+1,$$

which are congruent in some order to

$$0,\ 1,\ 2,\ \cdots,\ p-1,$$

modulo p. The same is true of the integers $m\beta$ and $m\gamma$, which range over the values

$$0,\ m,\ 2m,\ \cdots,\ (p-1)m.$$

Thus to every T' (class of transformations congruent to a given T modulo π) there corresponds an S' of H', *viz.*, the class of transformations

$$S \equiv \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \qquad \alpha'\delta' - \beta'\gamma' \equiv 1 \pmod{p},$$

where

$$\alpha' \equiv \alpha,\quad \beta' \equiv m\beta,\quad \gamma' \equiv m\gamma,\quad \delta' \equiv \delta \pmod{p}, \tag{1}$$

are integers reduced modulo p, and, conversely, to every such S' corresponds a T'. For we can solve the congruences

$$\begin{cases} \alpha \equiv \alpha',\quad m\beta \equiv \beta',\quad m\gamma \equiv \gamma',\quad \delta \equiv \delta' \pmod{p}, \\ \alpha \equiv \delta \equiv 1 \pmod{m}, \end{cases} \tag{2}$$

for integers α, $m\beta$, $m\gamma$, δ, reduced modulo π, and, since

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{p},$$

by the definition of S' and (2_1), while the same congruence holds modulo m by (2_2), it holds modulo π, and all transformations

$$T \equiv \begin{pmatrix} \alpha & m\beta \\ m\gamma & \delta \end{pmatrix} \pmod{\pi}$$

are in Γ. Further, this correspondence is one-to-one, since the solution of (1) in integers reduced modulo p is unique, and the same is true of the solution of (2) in integers reduced modulo π.

Also, given any two classes, T_1', T_2', of G', composed of all transformations

$$T_1 \equiv \begin{pmatrix} \alpha_1 & m\beta_1 \\ m\gamma_1 & \delta_1 \end{pmatrix}, \qquad T_2 \equiv \begin{pmatrix} \alpha_2 & m\beta_2 \\ m\gamma_2 & \delta_2 \end{pmatrix} \pmod{\pi},$$

respectively, where, from the definition of G,

$$\alpha_1 \equiv \delta_1 \equiv \alpha_2 \equiv \delta_2 \equiv 1 \pmod{m},$$

we have

$$\begin{aligned} T_3 &\equiv T_1 T_2 \\ &\equiv \begin{pmatrix} \alpha_1\alpha_2 + m^2\beta_1\gamma_2 & m\alpha_1\beta_2 + m\beta_1\delta_2 \\ m\gamma_1\alpha_2 + m\delta_1\gamma_2 & m^2\gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_3 & m\beta_3 \\ m\gamma_3 & \delta_3 \end{pmatrix} \pmod{\pi}. \end{aligned}$$

The product of two transformations S_1, S_2, belonging to the classes S_1', S_2', corresponding in the manner described to T_1', T_2', respectively, is

$$\begin{aligned} S_3 &\equiv S_1 S_2 \\ &\equiv \begin{pmatrix} \alpha_1' & \beta_1' \\ \gamma_1' & \delta_1' \end{pmatrix} \begin{pmatrix} \alpha_2' & \beta_2' \\ \gamma_2' & \delta_2' \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_1'\alpha_2' + \beta_1'\gamma_2' & \alpha_1'\beta_2' + \beta_1'\delta_2' \\ \gamma_1'\alpha_2' + \delta_1'\gamma_2' & \gamma_1'\beta_2' + \delta_1'\delta_2' \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_3' & \beta_3' \\ \gamma_3' & \delta_3' \end{pmatrix} \pmod{p}, \end{aligned}$$

where α_3', β_3', γ_3', δ_3' are the residues of α_3, $m\beta_3$, $m\gamma_3$, δ_3 modulo p, and hence S_3' is the class of H' corresponding to T_3' of G'.

Since p was any one of the p_i, we have

THEOREM II. *Each of the n sub-groups G_i' of the group Γ' of classes of transformations with determinant congruent to unity modulo π, viz., the group of all classes of Γ' composed of transformations which are congruent to identity modulo $m_i = \pi/p_i$, is isomorphic with the group H_i' of classes of transformations with determinant congruent to unity modulo p_i.*

5. We shall now prove the following

LEMMA. *For $i < n$, let T_1', T_2', $\cdots$, T_i' range over all the classes of G_1', G_2', $\cdots$, G_i', respectively. Then the products $T_1'T_2' \cdots T_i'$ are all distinct and form the sub-group J_i' of classes U_i' of transformations U_i of Γ which are congruent to identity modulo $l_i = \pi/(p_1p_2 \cdots p_i)$.*

This is true by definition when $i = 1$; i.e., $l_1 = m_1$. Suppose it true when the above i is replaced by $i - 1$. Then, first, the groups J_{i-1}' and G_i' have no class in common save that of the transformations congruent to identity (I) modulo π. For suppose any

$$U_{i-1}' = T_i'.$$

Then every

$$U_{i-1} \equiv T_i \pmod{\pi}.$$

But

$$T_i \equiv I \pmod{m_i},$$

and

$$U_{i-1} \equiv I \pmod{l_{i-1}},$$

and hence modulo p_i, a divisor of both l_{i-1} and π. Thus

$$T_i \equiv U_{i-1} \equiv I \pmod{p_i},$$

and, since m_i is prime to p_i and their product is π, we get

$$T_i \equiv I \pmod{\pi}.$$

Further, the products $U'_{i-1}T'_i$ are all distinct, where U'_{i-1}, T'_i range over all the classes of J'_{i-1}, G'_i, respectively. For, if such a product be equal to a second product (in which the letters are distinguished by an asterisk), i.e., if

$$U'_{i-1}T'_i = \overset{*}{U}'_{i-1}\overset{*}{T}'_i,$$

then every

$$U_{i-1}T_i \equiv \overset{*}{U}_{i-1}\overset{*}{T}_i \pmod{\pi},$$

and

$$\overset{*}{U}{}^{-1}_{i-1}U_{i-1} \equiv \overset{*}{T}_i T_i^{-1} \pmod{\pi},$$

whence each member of the last congruence is congruent to identity modulo π, and

$$\overset{*}{U}_{i-1} \equiv U_{i-1}, \qquad \overset{*}{T}_i \equiv T_i \pmod{\pi},$$

which implies

$$\overset{*}{U}'_{i-1} = U'_{i-1}, \quad \overset{*}{T}'_i = T'_i.$$

We note that $l_{i-1} = p_i l_i$, and that m_i is also divisible by l_i. Thus, since every

$$U_{i-1} \equiv I \pmod{l_{i-1}},$$

it satisfies the same congruence modulo l_i; and, similarly, every

$$T_i \equiv I \pmod{m_i},$$

and hence also modulo l_i. Therefore we have every product

$$U_{i-1}T_i \equiv I \pmod{l_i},$$

whence the products belong to J_i, and the product classes to J'_i.

Conversely, given the class U'_i of transformations

$$U_i \equiv \begin{pmatrix} \alpha & l_i\beta \\ l_i\gamma & \delta \end{pmatrix} \pmod{\pi},$$

where

$$\alpha \equiv \delta \equiv 1 \pmod{l_i},$$

we can find integers α_{i-1}, $l_{i-1}\beta_{i-1}$, $l_{i-1}\gamma_{i-1}$, δ_{i-1}, α_i, $m_i\beta_i$, $m_i\gamma_i$, δ_i, reduced modulo π such that

$$\alpha_{i-1}\alpha_i \equiv \alpha, \qquad \delta_{i-1}\delta_i \equiv \delta,$$

$$m_i\alpha_{i-1}\beta_i + l_{i-1}\beta_{i-1}\delta_i \equiv \beta, \qquad l_{i-1}\gamma_{i-1}\alpha_i + m_i\delta_{i-1}\gamma_i \equiv \gamma \pmod{\pi},$$

while

$$\alpha_{i-1} \equiv \delta_{i-1} \equiv 1 \pmod{l_{i-1}},$$

$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

so that, since

$$l_{i-1}m_i \equiv 0 \pmod{\pi},$$

we can factor each U_i, modulo π, into the product of

$$U_{i-1} \equiv \begin{pmatrix} \alpha_{i-1} & l_{i-1}\beta_{i-1} \\ l_{i-1}\gamma_{i-1} & \delta_{i-1} \end{pmatrix} \pmod{\pi},$$

and

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi}.$$

Thus all the classes U_i' of J_i' are expressible as products of classes U_{i-1}' of J_{i-1}', T_i' of G_i', and J_i' is obtainable by composition from J_{i-1}' and G_i'.

If i is taken to be n, the first part of the lemma is still true, thus all the products $T_1'T_2' \cdots T_n'$ are distinct, where T_1', T_2', $\cdots$, T_n' range over all the classes of G_1', G_2', $\cdots$, G_n', respectively. The order of Γ' is the product of the orders of the sub-groups G_1', G_2', $\cdots$, G_n'. For, given any class Θ' of transformations

$$\Theta \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{\pi},$$

where α, β, γ, δ are integers reduced modulo π, we can determine uniquely for each value of i a class S_i' of transformations

$$S_i \equiv \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \qquad \alpha_i\delta_i - \beta_i\gamma_i \equiv 1 \pmod{p_i},$$

where α_i, β_i, γ_i, δ_i are integers reduced modulo p_i, such that

$$\alpha_i \equiv \alpha, \quad \beta_i \equiv \beta, \quad \gamma_i \equiv \gamma, \quad \delta_i \equiv \delta \pmod{p_i};$$

and, conversely, given any set S_1', S_2', $\cdots$, S_n' we can solve

$$\alpha \equiv \alpha_i, \quad \beta \equiv \beta_i, \quad \gamma \equiv \gamma_i, \quad \delta \equiv \delta_i \pmod{p_i}$$

($i = 1, 2, \cdots, n$) for α, β, γ, δ, for which $\alpha\delta - \beta\gamma \equiv 1 \pmod{\pi}$ uniquely among the residues of integers modulo π, so that every set S_1', S_2', $\cdots$, S_n' determines a class Θ' of Γ'. Thus the order of Γ' is the product of the orders of the H_i', which equal those of the G_i', and we have

THEOREM III. *The total group Γ' of classes of transformations with determinant congruent to unity modulo π is obtainable by composition of the n sub-groups G_i', each composed of those classes of Γ' whose transformations are congruent to identity modulo $m_i = \pi/p_i$.*

6. Let $I(x, y)$ be any rational integral function with integral coefficients which is invariant under Γ modulo π. By Theorem I, this is equivalent to the invariance of $I(x, y)$ under the group Γ', i.e.,

$$I(\alpha x + \beta y, \gamma x + \delta y) \equiv I(x, y) \pmod{\pi},$$

for every set $\alpha, \beta, \gamma, \delta$, of integers reduced modulo π, for which

$$\alpha\delta - \beta\gamma \equiv 1 \pmod{\pi}.$$

In particular, $I(x, y)$ is invariant under every class T_i' of transformations

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi},$$
$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

of G_i' $(i = 1, 2, \cdots, n)$, and thus under the corresponding class S_i' of transformations

$$S_i \equiv \begin{pmatrix} \alpha_i' & \beta_i' \\ \gamma_i' & \delta_i' \end{pmatrix}, \qquad \alpha_i'\delta_i' - \beta_i'\gamma_i' \equiv 1 \pmod{p_i},$$

where

$$\alpha_i' \equiv \alpha_i, \quad \beta_i' \equiv m_i\beta_i, \quad \gamma_i' \equiv m_i\gamma_i, \quad \delta_i' \equiv \delta_i \pmod{p_i},$$

are integers reduced modulo p_i, of H_i' $(i = 1, 2, \cdots, n)$. By the isomorphism proved in Theorem II, when T_i' ranges over all the classes of G_i', S_i' ranges over all the classes of H_i', and thus $I(x, y)$ is invariant under all the transformations of H_i modulo p_i $(i = 1, 2, \cdots, n)$.

Conversely, suppose $I(x, y)$ is a rational integral invariant of any one of the H_i modulo p_i, i.e., invariant under every class S_i' of transformations

$$S_i \equiv \begin{pmatrix} \alpha_i' & \beta_i' \\ \gamma_i' & \delta_i' \end{pmatrix}, \qquad \alpha_i'\delta_i' - \beta_i'\gamma_i' \equiv 1 \pmod{p_i}.$$

Then, since we can solve

$$\alpha_i \equiv \alpha_i', \quad m_i\beta_i \equiv \beta_i', \quad m_i\gamma_i \equiv \gamma_i', \quad \delta_i \equiv \delta_i' \pmod{p_i},$$
$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

for integers α_i, $m_i\beta_i$, $m_i\gamma_i$, δ_i, reduced modulo π, $I(x, y)$ is invariant under the corresponding class T_i' of transformations

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi}$$

of G_i' modulo π, and thus, using again the isomorphism between G_i' and H_i', under the sub-group G_i' modulo π. Since, by Theorem III, the sub-groups G_i' $(i = 1, 2, \cdots, n)$ generate the total group Γ' modulo π, if any rational integral function with integral coefficients is invariant under every H_i' modulo p_i $(i = 1, 2, \cdots, n)$, it is an invariant of Γ', and hence of Γ, modulo π.

THEOREM IV. *A necessary and sufficient condition for the invariance of a rational integral function $I(x, y)$ with integral coefficients under the group Γ of transformations with determinant congruent to unity modulo $\pi = p_1 p_2 \cdots p_n$, is that $I(x, y)$ be invariant under every group H_i of transformations with determinant congruent to unity modulo p_i $(i = 1, 2, \cdots, n)$.*

7. As mentioned in § 2, a fundamental system of invariants of H_i modulo p_i is given by

$$L_i = x^{p_i} y - x y^{p_i},$$

$$Q_i = \frac{x^{p_i^2} y - x y^{p_i^2}}{L_i}.$$

Thus every rational integral invariant of H_i is congruent, modulo p_i, to a polynomial in L_i and Q_i with integral coefficients, or we may say that every invariant is of the form

$$F(L_i, Q_i) + p_i f(x, y),$$

where F and f are polynomials with integral coefficients.

By Theorem IV, if $I(x, y)$ is an invariant of Γ modulo π, it is an invariant of every H_i modulo p_i, thus it satisfies the equations

$$I(x, y) = F_i(L_i, Q_i) + p_i f_i(x, y), \tag{1}$$

$(i = 1, 2, \cdots, n)$. Since the greatest common divisor of the numbers m_i $(i = 1, 2, \cdots, n)$ is unity, there exist integers k_i such that

$$\sum_{i=1}^{n} k_i m_i = 1.$$

Multiplying each of the equations (1) by the corresponding $k_i m_i$, and adding, we have

$$I(x, y) = \sum_{i=1}^{n} k_i m_i F_i(L_i, Q_i) + \pi \sum_{i=1}^{n} k_i f_i(x, y),$$

whence

$$I(x, y) \equiv \sum_{i=1}^{n} k_i m_i F_i(L_i, Q_i) \pmod{\pi}.$$

As $k_i F_i(L_i, Q_i)$ is an invariant modulo p_i of H_i, we have, finally,

THEOREM V. *Every invariant of the group Γ of transformations with determinant congruent to unity modulo $\pi = p_1 p_2 \cdots p_n$ is a sum of expressions, each of which is the product of an $m_i = \pi/p_i$ by an invariant of the corresponding group H_i of transformations with determinant congruent to unity modulo p_i; and, conversely, every such product is an invariant of Γ modulo π by definition.*

DETERMINATION OF A SURFACE BY ITS CURVATURES AND SPHERICAL REPRESENTATION.

By W. C. Graustein.

1. **Introduction.**—The principal problem of this paper is to ascertain in how far a surface is determined by its curvatures and the directions of its normals,* or, stated more broadly, to find conditions, necessary and sufficient, that a definite, positive, differential quadratic form in the variables u, v and two functions, $K(u, v)$ and $\overline{K}(u, v)$, be respectively the third fundamental form and the total and mean curvatures of a surface, and to ascertain, when these conditions are fulfilled, the number of surfaces determined.†

The conditions that the given elements, in their prescribed capacities, determine at least one surface, though not easy to obtain in general form, are readily found when appropriate parameters are introduced.

Developable surfaces are evidently excluded by hypothesis. It is found that a non-developable surface is in general uniquely‡ determined by the prescribed elements. In the exceptional cases there is either a pair or a one-parameter family of surfaces. The surfaces of a pair or two surfaces of a family we shall call *cross-congruent.*

It is geometrically evident that the surfaces parallel respectively to those of a pair, or family, of cross-congruent surfaces and at the same algebraic distance, c, from them, also constitute a pair, or family, of cross-congruent surfaces. We shall call the two pairs, or families, *parallel.* If the ratio of the curvatures of the surfaces of the given pair, or family,

* This problem is to a certain extent analogous to the problem, proposed and solved by Bonnet, of determining the surfaces applicable to a given surface with preservation of both curvatures. Cf. Bonnet, "Mémoire sur la théorie des surfaces applicables sur une surface donneé," *Journal de l'École Polytechnique*, Vol. 42 (1867), pp. 72, foll.; also, Author, "Applicability with preservation of both curvatures," *Bull. Amer. Math. Soc.*, Vol. 30 (1924), where new results bearing on Bonnet's problem are derived by the methods developed in the present paper.

† Equivalent to the prescription of K and $\overline{K}$ is that of the principal normal curvatures, $1/r_1$ and $1/r_2$, provided that their order is left arbitrary. The sign of $\overline{K}$, or those of r_1 and r_2, are immaterial, since they depend on the orientation of the surface normal; however, if two surfaces satisfy the same prescribed conditions except perhaps that their mean curvatures are opposite in sign, we agree to orient their normals so that the mean curvatures will be equal.

‡ We adopt the usual convention of thinking of a surface as unique when it is determined except for rigid motions and reflections.

is $\overline{K}/K = r_1 + r_2$, that of the surfaces of the parallel pair, or family, is $\overline{K}/K - 2c$.

A family of associate minimal surfaces is a family of cross-congruent surfaces. So then is any parallel family. Families of these two types we shall call *families of associate surfaces for which* $\overline{K}/K = const.$ Both they and the function $\overline{K}/K$ play special rôles.

The lines of curvature of the surfaces of a pair are never represented on the sphere by isometric systems, whereas those of the surfaces of a family always are. In the latter case, the ∞^1 isometric systems on the sphere make constant angles with one another only in the case of associate surfaces for which $\overline{K}/K = \text{const.}$

The problem of determining whether or not two given surfaces are cross-congruent can be handled by the method used by Bonnet in treating the corresponding problem of Minding. The details are left to the reader. We note merely that, in imposing the condition that the curvatures be preserved, it is convenient to use, not K and $\overline{K}$, but K and $\overline{K}/K$. Our primary interest in this investigation consists in ascertaining under what conditions two surfaces are cross-congruent in a continuous infinity of ways. In stating these conditions we can assume that the surfaces are cross-congruent and are referred to the same parameters.

THEOREM 1. *Two cross-congruent surfaces are cross-congruent in a continuous infinity of ways if and only if their common curvatures,* K *and* $\overline{K}$, *are functionally dependent and the equation* $\overline{K}/K = const.$ *represents on the sphere an isometric family of geodesic parallels, or, if* $\overline{K}/K$ *is constant, the equation* $K = const.$ *represents such a family on the sphere.*

If $\overline{K}/K$ is not constant, the first condition of the theorem, namely, that K and $\overline{K}$ be functionally dependent, is, in the case of families of cross-congruent surfaces, by itself sufficient! Moreover, the differential system defining these families can be completely integrated and the point coordinates of the surfaces found by quadratures. When $\overline{K}/K$ is constant, no difficulties arise, inasmuch as cross-congruence, in the special case of minimal surfaces, is identical with applicability.

The second half of the paper deals with the problem from the point of view of parallel maps.* It is evident that either of two cross-congruent surfaces can be rotated and then, if necessary, reflected in a point, so that the normals in corresponding points will be parallel and similarly oriented. Hence *the problem of finding the surfaces cross-congruent to a given surface, if any exist, is identical with the problem of determining the surfaces in point correspondence with the given surface by parallel normals so that the total and*

* Cf. Author, "Parallel Maps of Surfaces," *Trans. Amer. Math. Soc.*, Vol. 23 (1922), pp. 298–332. This paper will be referred to as "Reference A."

mean curvatures in corresponding points are respectively equal. A parallel map of this type we shall call *cross-congruent.* Of course, it is assumed that the map is not equivalent to a rigid motion or a reflection.

A parallel map is non-parabolic or parabolic, according as there exist on the surfaces basic, i.e., corresponding, conjugate systems, or merely basic families of asymptotic lines.

As a fundamental result we note that there is at most one surface cross-congruent to a given surface by a parallel map for which the basic conjugate system, or family of asymptotic lines, is prescribed, save when the given surface is minimal and the prescribed conjugate system consists of its generators. In other words, the basic systems of the parallel maps of one of ∞^1 cross-congruent surfaces on the others are all distinct save in the case of a family of associate minimal surfaces.

A non-parabolic map is cross-congruent if and only if the normal curvatures of the two surfaces in the basic conjugate directions are equal crosswise, but not also directly, that is, if and only if each of these curvatures for the one surface is equal to the non-corresponding one, but not also to the corresponding one, on the other surface. Herein lies the motivation of the term "cross-congruent"; if the curvatures were directly equal, the surfaces would be congruent. Moreover, in a map of the type in question, the point invariants of the basic conjugate systems are equal crosswise.

A parabolic map is cross-congruent only when the two surfaces are suitably related ruled surfaces whose secondary asymptotic lines cut equal segments from the rulings.

Various particular questions of interest are taken up, for example, surfaces of revolution, surfaces of constant total curvature or of constant mean curvature, not zero, in the first part of the paper, and in the second part, special cases of cross-congruent non-parabolic maps, including those of translation surfaces, of associate surfaces of Bianchi, and those in which the basic conjugate systems have equal point invariants.

I. DIRECT TREATMENT.

2. **Existence theorems.**—Let the given definite, positive, differential quadratic form be

$$\mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2, \tag{1}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are real, analytic functions of the real variables u, v, and let the given functions, $K(u, v)$, $\overline{K}(u, v)$, be also real and analytic.

An evident necessary condition that there exist a surface, $x = x(u, v)$:

$$x_1 = x_1(u, v), \qquad x_2 = x_2(u, v), \qquad x_3 = x_3(u, v), \tag{2}$$

whose total and mean curvatures are K and $\overline{K}$ and whose third fundamental

form is (1), is that the curvature of (1) be unity. We assume henceforth that this condition is fulfil ed.

If a surface (2) is determined by the prescribed conditions, the coefficients e, f, g of its second fundamental form satisfy a system of four equations, namely, the two differential equations of Codazzi, formed with respect to (1), and the two finite equations,

$$\frac{\mathcal{E}\mathcal{G} - \mathcal{F}^2}{eg - f^2} = K, \qquad \frac{\mathcal{E}g - 2\mathcal{F}f + \mathcal{G}e}{eg - f^2} = \overline{K}. \tag{3}$$

This system of equations we shall call the system A.

Conversely, if for given $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, K, $\overline{K}$, the system A is compatible, each solution, e, f, g, determines a unique surface (2).

If we set $\mathcal{F} = 0, f = 0$, that is, if we choose as the parametric curves of the given spherical representation an orthogonal system which will represent the lines of curvature on at least one of the required surfaces, if any exist, the Codazzi equations become

$$\frac{\partial e}{\partial v} = \frac{1}{2}\left(\frac{e}{\mathcal{E}} + \frac{g}{\mathcal{G}}\right)\frac{\partial \mathcal{E}}{\partial v}, \qquad \frac{\partial g}{\partial u} = \frac{1}{2}\left(\frac{e}{\mathcal{E}} + \frac{g}{\mathcal{G}}\right)\frac{\partial \mathcal{G}}{\partial u}, \tag{4}$$

and conditions (3) can be replaced by

$$\frac{e}{\mathcal{E}} = r_1, \qquad \frac{g}{\mathcal{G}} = r_2,$$

where r_1 and r_2 are, in an arbitrary order, the prescribed values of the principal radii of curvature corresponding to those of K and $\overline{K}$. This system A is compatible if and only if*

$$\frac{\partial \log \mathcal{E}}{\partial v} = -\frac{1}{N}\frac{\partial r_1}{\partial v}, \qquad \frac{\partial \log \mathcal{G}}{\partial u} = \frac{1}{N}\frac{\partial r_2}{\partial u}, \tag{5}$$

where

$$2M = r_1 + r_2 = \frac{\overline{K}}{K}, \qquad 2N = r_1 - r_2. \tag{6}$$

THEOREM 2. *A necessary and sufficient condition that there exist at least one surface having a given linear element of the spherical representation and given curvatures is that there exist on the sphere an orthogonal system such that, when it is parametric, equations* (5), *where* r_1 *and* r_2 *are the prescribed principal radii of curvature in a suitable order, are satisfied.*

We assume henceforth that on the sphere the orthogonal system in question is parametric and that equations (5) are fulfilled. The surface then determined by $\mathcal{E}$, $\mathcal{F} = 0$, $\mathcal{G}$ and

$$e = r_1\mathcal{E}, \qquad f = 0, \qquad g = r_2\mathcal{G}, \tag{7}$$

* The trivial case, in which r_1 and r_2 are equal and therefore constant and the surface is a sphere, is here and henceforth excluded.

we denote by S. Its lines of curvature are represented by the orthogonal system on the sphere.

If there exists a second surface, S', fulfilling the given conditions, then

$$e'g' - f'^2 = eg, \qquad \mathcal{E}g' + \mathcal{G}e' = \mathcal{E}g + \mathcal{G}e.$$

Hence

$$\frac{e - e'}{\mathcal{E}} = \frac{g' - g}{\mathcal{G}}.$$

The common value of these two ratios can be shown to be $2N \sin^2 \varphi$, where φ is a determination of the angle between the curves on the sphere representing the lines of curvature on S and S'. In terms of φ we can express e', f', g':

$$(8) \qquad e' = e - 2\mathcal{E}N \sin^2 \varphi, \qquad f' = \sqrt{\mathcal{E}\mathcal{G}}\, N \sin 2\varphi, \qquad g' = g + 2\mathcal{G}N \sin^2 \varphi.$$

It remains to determine the parameter φ so that e', f', g' satisfy the Codazzi equations for S'. These, by virtue of the corresponding equations (4) for S, can be written in the forms:

$$\frac{\partial(e' - e)}{\partial v} = f' \frac{\partial}{\partial u} \log\left(f' \sqrt{\frac{\mathcal{G}}{\mathcal{E}}}\right), \qquad \frac{\partial(g' - g)}{\partial u} = f' \frac{\partial}{\partial v} \log\left(f' \sqrt{\frac{\mathcal{E}}{\mathcal{G}}}\right).$$

Substituting for $e' - e, f', g' - g$ their values as given by (8) and reducing, we obtain two equations linear in the partial derivatives of $\cot \varphi$. Thence we find

$$(9) \qquad \frac{\partial \cot \varphi}{\partial u} = A \cot \varphi - B \sqrt{\frac{\mathcal{E}}{\mathcal{G}}}, \qquad \frac{\partial \cot \varphi}{\partial v} = B \cot \varphi + A \sqrt{\frac{\mathcal{G}}{\mathcal{E}}},$$

where

$$(10) \qquad A = \frac{1}{N} \frac{\partial M}{\partial u} = \frac{\partial \log \mathcal{G}N}{\partial u}, \qquad B = -\frac{1}{N} \frac{\partial M}{\partial v} = \frac{\partial \log \mathcal{E}N}{\partial v}.$$

But equations (9) are compatible if and only if

$$P \cot \varphi = Q,$$

where

$$(11) \qquad \begin{aligned} P &= \frac{\partial A}{\partial v} - \frac{\partial B}{\partial u} = \frac{\partial^2}{\partial u \partial v} \log \frac{\mathcal{G}}{\mathcal{E}}, \\ Q &= \frac{\partial}{\partial u}\left(A \sqrt{\frac{\mathcal{G}}{\mathcal{E}}}\right) + \frac{\partial}{\partial v}\left(B \sqrt{\frac{\mathcal{E}}{\mathcal{G}}}\right) - A^2 \sqrt{\frac{\mathcal{G}}{\mathcal{E}}} - B^2 \sqrt{\frac{\mathcal{E}}{\mathcal{G}}}. \end{aligned}$$

If $P = 0$ and $Q = 0$, equations (9) have ∞^1 solutions. If $P \neq 0$ and $\cot \varphi = Q/P$ satisfies (9), these equations have a single solution. In all other cases, no solution exists.

THEOREM 3. *If the conditions of Theorem 2 are satisfied, there is in*

general but one surface determined by the prescribed elements. In the exceptional cases there are two or a one-parameter family: two, if $P \neq 0$ *and* $\cot\varphi = Q/P$ *satisfies equations* (9); *a one-parameter family, if* $P = 0$ *and* $Q = 0$.

It follows, then, that a non-developable surface is in general uniquely determined by the linear element of its spherical representation and its total and mean curvatures.

3. **Families of cross-congruent surfaces.**—Since, in this case, $P = 0$, we conclude from (11) that the orthogonal system on the sphere is isometric. Assuming that the original parameters were isometric:

$$\mathcal{E} = \mathcal{G} = \lambda,$$

we find from (11), using the second values of A and B in (10), that

$$Q = -\lambda N\left(\frac{\partial^2}{\partial u^2}\frac{1}{\lambda N} + \frac{\partial^2}{\partial v^2}\frac{1}{\lambda N}\right).$$

Hence the vanishing of Q is the condition that $1/\lambda N$ be a harmonic function.

THEOREM 4. *The prescribed elements determine a family of cross-congruent surfaces if and only if there exist isometric parameters on the sphere such that, when they are introduced, equations* (5), *where* $\mathcal{E} = \mathcal{G} = \lambda$, *are satisfied and the reciprocal of* $\lambda(r_1 - r_2)$ *is a harmonic function.*

The second condition can be replaced by a condition on the curvatures alone, obtained by substituting in $Q = 0$ the values of A and B first given in (10).

Equations (9) can now be written in the forms:

$$\frac{\partial}{\partial u}\frac{\cot\varphi}{\lambda N} = \frac{\partial}{\partial v}\frac{1}{\lambda N}, \qquad \frac{\partial}{\partial v}\frac{\cot\varphi}{\lambda N} = -\frac{\partial}{\partial u}\frac{1}{\lambda N}.$$

If one function, μ, to which $1/\lambda N$ is conjugate, is found by a quadrature, the general solution of these equations is

$$\cot\varphi = \lambda N(\mu + k). \tag{12}$$

The arbitrary constant k is the parameter of the family; when finite, it determines the surfaces S' and in becoming infinite it defines S.

If the point coördinates, $\zeta = \zeta(u, v)$, of the given spherical representation are known, those of the surfaces S and S' can be found in the usual way by quadratures. We have, namely,

$$x = -\int r_1\zeta_u du + r_2\zeta_v dv, \tag{13}$$

$$x' = x + 2\int N\sin^2\varphi[(\zeta_u - \cot\varphi\zeta_v)du - (\cot\varphi\zeta_u + \zeta_v)dv], \tag{14}$$

where $\cot\varphi$ is given by (12).

It is evident that the properties which we have established for S hold also for the surfaces S'. In particular:

THEOREM 5. *The lines of curvature on ∞^1 cross-congruent surfaces are represented by isometric systems on the sphere.*

Though the correspondence of cross-congruence between two surfaces of a family never itself reduces to that of actual congruence, it is still conceivable that the surfaces be congruent. This is, however, not in general the case. For it can readily be shown that the two surfaces are congruent if and only if there exist for each isometric parameters, corresponding to the isometric systems of Theorem 5, such that the prescribed linear element of the sphere and the total and mean curvatures are the same expressions in the one pair of parameters as they are in the other.*

The surfaces S and S' correspond by a parallel map (cf. Introduction). The basic conjugate systems of this map are defined by

$$\cot\varphi(r_1du^2 - r_2dv^2) + (r_1 + r_2)dudv = 0. \tag{15}$$

Two equations (15), formed for two different values of the parameter k involved in $\cot\varphi$, are always distinct, unless $r_1 + r_2 = 0$.

THEOREM 6. *The ∞^1 basic conjugate systems on an arbitrary one of a family of cross-congruent surfaces in its parallel maps on the others are all distinct, except in the case of associate minimal surfaces, when they all coincide in the minimal lines.*

The map of S on S' is parabolic if and only if the discriminant of (15) vanishes, i.e., if and only if

$$\cot^2\varphi = -\frac{\overline{K}^2}{4K}.$$

If $K > 0$, this condition is never fulfilled; if $K < 0$, it can hold for at most one value of the parameter k. Thus, each surface of a family is mapped parabolically on at most one, and in general on none, of the others.

4. **Associate surfaces for which $\overline{K}/K$ = const.**—It is well known that two minimal surfaces are applicable only if they are associate or congruent to associates. But two associate minimal surfaces have the same spherical representation and are therefore cross-congruent. Conversely, if two minimal surfaces, S and S', are cross-congruent, the correspondence between them is isometric. This is an immediate consequence of

$$KE - \overline{K}e + \mathcal{E} = 0, \qquad KE' - \overline{K}e' + \mathcal{E} = 0, \tag{16}$$

and the corresponding equations, since $\overline{K} = 0$.

* The surface S' is congruent to S only if suitable isometric parameters, u', v', of the type in question, can be introduced for S', so that λ' ($= \mathcal{E}' = \mathcal{G}'$), $e', f' = 0, g'$ will be the same functions of u', v' as $\lambda, e, f = 0, g$ are of u, v, or, since $e'/\lambda', g'/\lambda'$ and $e/\lambda, g/\lambda$ are the principal radii of curvature, so that λ' will be the same function as λ and r_1' and r_2' the same functions as r_1 and r_2 (or r_2 and r_1). But the latter condition, in either form, is equivalent to demanding that K' and $\overline{K}'$ be the same functions as K and $\overline{K}$.

THEOREM 7. *Two minimal surfaces are cross-congruent if and only if they are applicable; this is the only case in which a cross-congruent correspondence is isometric.*

The point coördinates of a real minimal surface and the surfaces parallel to it are

$$x = U(u) + \overline{U}(\bar{u}) - \frac{c}{2}\zeta, \tag{17}$$

where c is an arbitrary real constant and $U(u)$ stands for the triple of functions,

$$\int \frac{1-u^2}{2}\psi(u)du, \qquad i\int \frac{1+u^2}{2}\psi(u)du, \qquad \int u\psi(u)du,$$

$\overline{U}(\bar{u})$ for the triple of conjugate complex functions of the conjugate variable $\bar{u}$, and ζ for the point coördinates of the common spherical representation.

The surfaces

$$x = e^{\alpha i}U(u) + e^{-\alpha i}\overline{U}(\bar{u}) - \frac{c}{2}\zeta, \tag{18}$$

associate to, and including, a given surface (17) constitute a complete family of cross-congruent surfaces (cf. Introduction).

For the associate surfaces (18), $\overline{K}/K = 2M$ has the constant value c. But then, by (10), λN is constant, and hence, by (12), so also is the angle φ. Conversely, if φ is constant, $\overline{K}/K$ is constant.

THEOREM 8. *Only in the case of a family of associate surfaces for which $\overline{K}/K$ is constant do the orthogonal systems on the sphere representing the lines of curvature of the surfaces of the family make constant angles with one another.*

If as S we take the surface (17) and as S' the surface (18), it follows from the theory of associate minimal surfaces that $\varphi = \pm\,\alpha/2$. Consequently, for the surfaces (17) and

$$x = -\,U(u) - \overline{U}\,(\bar{u}) - \frac{c}{2}\zeta, \tag{19}$$

$\varphi = \pm\,\pi/2$. These surfaces are cross-congruent, provided $c \neq 0$, and their lines of curvature correspond. Moreover, this is evidently the only case in which the lines of curvature of two cross-congruent surfaces correspond, or, what is equivalent, the only case in which equations (5) are satisfied when r_1 and r_2 are taken in both orders.

THEOREM 9. *The lines of curvature of two cross-congruent surfaces correspond if and only if they are non-minimal associate surfaces of the types* (17) *and* (19).

By Theorem 7, a minimal surface is cross-congruent to itself, or to each of its associates, in a continuous infinity of ways only if it admits a con-

tinuous deformation into itself. The facts concerning minimal surfaces with this property are well known,* and, when generalized to the case $c \neq 0$, lead to the following results.

THEOREM 10. *A surface* (17), *for which* $\overline{K}/K$ *is constant, is cross-congruent to itself, or to each of its associates, in a continuous infinity of ways if and only if the functions* $\psi(u)$ *and* $\overline{\psi}(\bar{u})$ *are of the forms:* †

$$\psi(u) = au^b, \qquad \overline{\psi}(\bar{u}) = \bar{a}\bar{u}^b. \tag{20}$$

5. **Equations defining all families of cross-congruent surfaces.**—Equations (5), or the equivalent equations (10), can be rewritten as

$$\frac{\partial M}{\partial u} = N\frac{\partial \log \lambda N}{\partial u}, \qquad \frac{\partial M}{\partial v} = -N\frac{\partial \log \lambda N}{\partial v}. \tag{21}$$

These equations are compatible if and only if

$$\frac{\partial}{\partial u}\left(N\frac{\partial \log \lambda N}{\partial v}\right) + \frac{\partial}{\partial v}\left(N\frac{\partial \log \lambda N}{\partial u}\right) = 0. \tag{22}$$

If to (22) we adjoin the condition that $1/\lambda N$ be a harmonic function:

$$\frac{\partial^2}{\partial u^2}\frac{1}{\lambda N} + \frac{\partial^2}{\partial v^2}\frac{1}{\lambda N} = 0, \tag{23}$$

and the Gauss equation for the sphere,

$$\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2} + 2\lambda = 0, \tag{24}$$

we have three equations in λ and N, whose simultaneous solutions yield all the families of cross-congruent surfaces. For, M is determined from (21) except for an additive constant, cot φ can be found from (12), and finally, x and x' from (13) and (14). Only quadratures are necessary, if ζ is known.

As might have been expected, each continuum of surfaces constituting ∞^1 parallel families appears as a unit, corresponding to a pair of particular solutions of the three equations.

THEOREM 11. *A particular pair of simultaneous solutions,* λ, N, *of the equations* (22), (23), (24) *determines* ∞^1 *parallel families of cross-congruent*

* Cf. Eisenhart, "Differential Geometry," p. 328.

† The theorem is not strictly true, for if $b = -2$, the surface is a helicoid (or in a limiting case, a catenoid of revolution) or a surface parallel to it, and is not cross-congruent to itself in ∞^1 ways, but congruent, the path curves being helices (circles). But the surface is cross-congruent to each of its associates in ∞^1 ways and not congruent to any of them.

If $b \neq -2$, the surface is cross-congruent to itself, and to each of its associates, in a continuous infinity of ways. Here, however, the surface is actually congruent to each associate, and we have an example of ∞^1 cross-congruent surfaces, each two of which are actually congruent.

surfaces. The point coördinates of the surfaces can be found by quadratures, provided those of the sphere are known.

6. **Solution of the equations when the curvatures are functionally dependent.**—A functional relation exists between K and $\overline{K}$ when and only when a similar relation exists between M and N. But M and N are functionally dependent, by virtue of (21), if and only if

$$\frac{\partial N}{\partial u}\frac{\partial \log \lambda N}{\partial v} + \frac{\partial N}{\partial v}\frac{\partial \log \lambda N}{\partial u} = 0. \tag{25}$$

But then (22) becomes

$$\frac{\partial^2 \log \lambda N}{\partial u \partial v} = 0,$$

and

$$\frac{1}{\lambda N} = mU(u)V(v), \tag{26}$$

where m is an arbitrary constant, not zero.

Three cases arise, according to the nature of the functions $U(u)$, $V(v)$.

A. If both functions are constant, λN is constant and hence, by (21), S and S' are surfaces for which $\overline{K}/K$ is constant.

B. If just one of the functions is constant, λN is a function of but one of the parameters, say u. But then, by (21) and (25), λ, M, and N are all functions of u alone and hence S is a surface of revolution. Conversely, if S is a surface of revolution, λ, M, and N depend on but one of the parameters, and hence λN is constant or depends merely on this parameter. Consequently, all the surfaces of revolution belonging to families of cross-congruent surfaces are covered by this case or a limiting form of it of type A.

Since $\lambda = \lambda(u)$, the general solution of (24) is $\lambda = c^2 \operatorname{sech}^2 c(u + d)$. Replacing $c(u + d)$ and cv by new parameters u, v, we reduce λ to

$$\lambda = \operatorname{sech}^2 u.$$

We can take, then, as the spherical representation,

$$\zeta_1 = \operatorname{sech} u \cos v, \qquad \zeta_2 = \operatorname{sech} u \sin v, \qquad \zeta_3 = \tanh u. \tag{27}$$

From (23), we find that

$$\lambda N = \frac{1}{au + b}, \qquad a^2 + b^2 \neq 0.$$

Thus we have case B itself or its limiting form of type A, according as $a \neq 0$ or $a = 0$.

It is now readily shown that

$$r_1 = \frac{2 \cosh^2 u}{au + b} - \int \frac{\sinh 2u}{au + b}\, du, \qquad r_2 = -\int \frac{\sinh 2u}{au + b}\, du,$$

where the integral is arbitrary, i.e., contains its own additive constant. Hence the finite equations (13) of S become

$$(28) \qquad x_1 = -r_2\zeta_1, \qquad x_2 = -r_2\zeta_2, \qquad x_3 = -\int r_1 \operatorname{sech}^2 u\,du.$$

THEOREM 12. *The surfaces of revolution* (28) *are the only surfaces of revolution which belong to families of cross-congruent surfaces. They consist of surfaces parallel to and including the catenoids and surfaces closely related to these.*

In connection with the first part of the theorem, we note that a surface of revolution can never belong to a pair of cross-congruent surfaces. For, since $P \neq 0$ in this case, neither system of curves on the sphere representing the lines of curvatures on the surfaces of a pair is isometric. It follows that every surface of revolution, other than those defined by (28), is uniquely determined by its spherical representation and curvatures.

To establish the last part of the theorem, we note that the surfaces (28), when $a = 0$, must be catenoids or surfaces parallel to them, since the only minimal surfaces of revolution are the catenoids. It follows, also, that in the case, $a = 0$, the families of cross-congruent surfaces are precisely the families parallel to and including those of the catenoids and their associate helicoids, i.e., the families of Theorem 10 for $b = -2$.

A family consisting of a catenoid and the associate helicoids contains just the one surface of revolution. But a parallel family contains two surfaces of revolution, parallel to and at equal distances from a catenoid. This is the only case of this sort, for if two surfaces of revolution are cross-congruent, their lines of curvature must correspond and Theorem 9 applies.

THEOREM 13. *Two surfaces of revolution are cross-congruent only if they are parallel to the same catenoid at equal distances from it.*

C. In the general case, when neither of the functions U, V in (26) is constant, equations (25) and (21) are readily solved for N and M. The resulting values of λ, M, and N, expressed in terms of the new parameters,

$$w = \log U - \log V, \qquad t = \log U + \log V,$$

are

$$(29) \qquad \lambda = \frac{1}{e^t\psi(w)}, \qquad mM = -\int \psi(w)dw, \qquad mN = \psi(w),$$

where $\psi(w)$ is an unknown function. Thus w is a single parameter upon which both the curvatures depend.

It remains to determine, from (23) and (24), the functions $U(u)$, $V(v)$, $\psi(w)$. By virtue of (26), (23) becomes

$$\frac{U''}{U} = -\frac{V''}{V} = k,$$

where the constant k can be taken as non-negative.

When $k = 0$, $U = au + a_0$, $V = bv + b_0$, where $ab \neq 0$. We can take $a = b$, because of the presence of the constant m in (26), and then change to new isometric parameters so that

Ia $$U = u, \qquad V = v.$$

When $k \neq 0$, it is convenient to distinguish three cases, which can be defined without loss of generality by the following pairs of values for U and V:

Ib $$U = \sinh u, \qquad V = \sin v,$$

II $$U = \cosh u, \qquad V = \sin v,$$

III $$U = e^u, \qquad V = \sin v.$$

For each of the four pairs of values for U and V, equation (24) becomes an ordinary differential equation for the determination of $\psi(w)$, namely,

$$\frac{d}{dw}\left(L\frac{d\log\psi}{dw}\right) - L = \frac{1}{\psi}, \tag{30}$$

where L has in the several cases the values:

$$\text{I: } L = \cosh w, \qquad \text{II: } L = \sinh w, \qquad \text{III: } L = \tfrac{1}{2}e^w.$$

We shall give presently the explicit solutions of these equations. Thereby we shall have determined all the families of cross-congruent surfaces under the present case. Moreover, we shall show that here, as in the previous cases, the point coördinates of the spherical representation can be found without recourse to the usual Riccati equation, and hence that the point coördinates of the surfaces can be found by quadratures (cf. Th. 11).

THEOREM 14. *The point coördinates of every family of cross-congruent surfaces whose curvatures are functionally dependent can be found by quadratures.*

Since

I $$\Delta_1 w = 2\psi\cosh w, \qquad \Delta_2 w = 2\psi\sinh w,$$

II $$\Delta_1 w = 2\psi\sinh w, \qquad \Delta_2 w = 2\psi\cosh w,$$

III $$\Delta_1 w = \psi e^w, \qquad \Delta_2 w = \psi e^w,$$

where the differential parameters are formed with respect to the linear element, $\lambda(du^2 + dv^2)$, of the sphere, the family of curves $w = \text{const.}$ ($\overline{K}/K = \text{const.}$) on the sphere is an isometric family of geodesic parallels. Since this is true also in case B, we conclude, by Theorem 1, the following:

THEOREM 15. *Two surfaces of a family of cross-congruent surfaces whose curvatures are functionally related, but of variable ratio, are always cross-congruent in a continuous infinity of ways.*

In other words, when $\overline{K}/K$ is not constant, the first condition of Theorem 1 is in itself sufficient.

For the isometric system on the sphere, consisting of the family w = const. and its orthogonal trajectories, we now introduce isometric parameters, $\bar{u}$, $\bar{v}$, computed in the usual way:

$$\text{Ia} \quad \bar{u} = 2\tan^{-1} e^{w} = 2\tan^{-1}\frac{u}{v}, \qquad \bar{v} = \log(u^2+v^2),$$

$$\text{Ib} \quad \bar{u} = 2\tan^{-1} e^{w} = 2\tan^{-1}\frac{\sinh u}{\sin v}, \qquad \bar{v} = \log\frac{\cosh u - \cos v}{\cosh u + \cos v},$$

$$\text{II} \quad \bar{u} = \log\tanh\frac{w}{2} = \log\frac{\cosh u - \sin v}{\cosh u + \sin v}, \qquad \bar{v} = 2\tan^{-1}\frac{\sinh u}{\cos v},$$

$$\text{III} \quad \bar{u} = e^{-w} = e^{-u}\sin v, \qquad \bar{v} = e^{-u}\cos v.$$

Returning to equation (30), we introduce instead of w the isometric parameter $\bar{u}$ and instead of ψ the function ξ, where

$$\text{I: } \xi = \psi\sin\bar{u}, \qquad \text{II: } \xi = -\psi\sinh\bar{u}, \qquad \text{III: } \xi = \tfrac{1}{2}\psi\bar{u}.$$

In all three cases (30) reduces to the equation,

$$\frac{d^2\log\xi}{d\bar{u}^2} = \frac{1}{\xi},$$

whose general solution is

$$2a^2\xi = \cosh^2(a\bar{u}+b).$$

Hence the general solutions of (30) are

$$\text{(31)} \quad \begin{aligned} &\text{I} && 2a^2\psi = \csc\bar{u}\cosh^2(a\bar{u}+b),\\ &\text{II} && 2a^2\psi = -\operatorname{csch}\bar{u}\cosh^2(a\bar{u}+b),\\ &\text{III} && a^2\psi = \frac{1}{\bar{u}}\cosh^2(a\bar{u}+b). \end{aligned}$$

We are now able to express the linear element of the sphere in terms of $\bar{u}$, $\bar{v}$. We find, in all cases,

$$d\sigma^2 = a^2\operatorname{sech}^2(a\bar{u}+b)(d\bar{u}^2+d\bar{v}^2).$$

Corresponding point coördinates of the sphere can be obtained directly by replacing u and v in (27) by $a\bar{u}+b$ and $a\bar{v}$. Finally, the values, in the four cases, of M and N, and hence of r_1 and r_2, in terms of $\bar{u}$, $\bar{v}$, can be readily computed from (29) and (31). We have thus completed the proof of Theorem 14.

From the formulas for r_1 and r_2 in this and the previous case, it is evident that K is never constant and that $\overline{K}$ is never constant, unless zero.

THEOREM 16. *There can be at most a pair of cross-congruent surfaces when* (a) *the total curvature is constant, or* (b) *the mean curvature is constant, not zero, or* (c) *the curvatures are connected by a linear relation,* $aK + b\overline{K} = c$, *where* a, b, c *are constants,* $c \neq 0$.

The third case is that of surfaces parallel to those of the first two.

7. **Pairs of cross-congruent surfaces.**—If for A and B we take the second values in (10), namely,

$$A = \frac{\partial \log \mathcal{G}N}{\partial u}, \qquad B = \frac{\partial \log \mathcal{E}N}{\partial v},$$

equations (5) can be rewritten in the forms:

$$\frac{\partial M}{\partial u} = AN, \qquad \frac{\partial M}{\partial v} = -BN, \tag{32}$$

whence

$$\frac{\partial(BN)}{\partial u} + \frac{\partial(AN)}{\partial v} = 0. \tag{33}$$

To equation (33) we adjoin the Gauss equation for the sphere and the conditions that $\cot \varphi = Q/P$, where P and Q are given by (11) and $P \neq 0$, satisfy equations (9). We thus have four equations in the three unknown functions, $\mathcal{E}$, $\mathcal{G}$, N. Their simultaneous solutions determine all the pairs of cross-congruent surfaces; for M is then given by (32) and φ directly by $\cot \varphi = Q/P$. In particular, a single set of solutions defines ∞^1 parallel pairs of cross-congruent surfaces.

The two systems of curves on the sphere representing the lines of curvature on the two surfaces never cut under a constant angle, for the assumption that Q/P is constant leads to a contradiction. Moreover, since $P \neq 0$, neither system of curves on the sphere is isometric. A surface cannot belong both to a pair and a family of cross-congruent surfaces.

The correspondence by parallel normals between two surfaces of a pair is in general non-parabolic, as in the case of two surfaces of a family. In fact, the condition that the map be parabolic is the same in the two cases (cf. § 3).

II. TREATMENT BY THE THEORY OF PARALLEL MAPS.

8. **Non-parabolic parallel maps.**—Let S and S':

$$x = x(u, v), \qquad x' = x'(u, v),$$

be two non-minimal* surfaces in one-to-one point correspondence by parallel normals so that corresponding points have the same curvilinear

* The facts concerning minimal surfaces have been fully established in § 4.

coördinates. Assume that the map is not parabolic and that the basic conjugate systems are parametric. Then scalar functions, $\lambda(u, v)$, $\mu(u, v)$, exist so that*

$$x'_u = \lambda x_u, \qquad x'_v = \mu x_v. \tag{34}$$

The invariant I† of the map is

$$I = \frac{\lambda}{\mu},$$

and‡

$$E' = \lambda^2 E, \quad F' = \lambda\mu F, \quad G' = \mu^2 G, \quad D' = \lambda\mu D,$$
$$e' = \lambda e, \quad f' = f = 0, \quad g' = \mu g, \quad d'^2 = \lambda\mu d^2.$$

The surfaces S and S' are cross-congruent if and only if

$$\lambda\mu = 1, \qquad (\lambda - 1)Eg + (\mu - 1)Ge = 0, \qquad \lambda \neq \mu.$$

If R_1 and R_2 denote the radii of normal curvature of S in the directions of the curves $v =$ const. and $u =$ const. respectively, and R'_1 and R'_2 have the same significance for S', these conditions are equivalent to

$$\lambda R_1 - R_2 = 0, \qquad R_1 - \mu R_2 = 0, \qquad R_1 \neq R_2, \tag{35}$$

which, by virtue of the relations, $R'_1 = \lambda R_1$, $R'_2 = \mu R_2$, are in turn equivalent to

$$R'_1 = R_2, \qquad R'_2 = R_1, \qquad R_1 \neq R_2.$$

THEOREM 17. *A necessary and sufficient condition that two non-minimal surfaces mapped non-parabolically by parallel normals be cross-congruent is that their radii of normal curvature in the basic conjugate directions be equal crosswise.*

The invariant of the map is

$$I = \left(\frac{R_2}{R_1}\right)^2. \tag{36}$$

From (34) follows

$$(\lambda - \mu)x_{uv} + \lambda_v x_u - \mu_u x_v = 0, \tag{37}$$

whence

$$(\lambda - \mu)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \lambda_v = 0, \qquad (\lambda - \mu)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \mu_u = 0. \tag{38}$$

Introduce the conditions (35) for cross-congruence and these equations

* Cf. Reference A, p. 302.

† Ibid., p. 300.

‡ $D^2 = EG - F^2$, $d^2 = eg - f^2$.

become

$$(39)\qquad \frac{\partial \log R_1/R_2}{\partial v} = \left[1 - \left(\frac{R_1}{R_2}\right)^2\right]\left\{\begin{matrix}12\\1\end{matrix}\right\},$$
$$\frac{\partial \log R_2/R_1}{\partial u} = \left[1 - \left(\frac{R_2}{R_1}\right)^2\right]\left\{\begin{matrix}12\\2\end{matrix}\right\},$$

or

$$(40)\qquad \frac{\partial}{\partial v}\log\frac{R_1^2}{R_1^2 - R_2^2} = 2\left\{\begin{matrix}12\\1\end{matrix}\right\}, \qquad \frac{\partial}{\partial u}\log\frac{R_2^2}{R_1^2 - R_2^2} = 2\left\{\begin{matrix}12\\2\end{matrix}\right\},$$

where

$$R_1 \neq R_2.$$

Conversely, given a non-minimal surface S referred to a conjugate system so that equations (40) are satisfied. Then $\lambda = R_2/R_1$ and $\mu = R_1/R_2$ satisfy (38) and hence (37), and therefore equations (34) are integrable. Thus there is determined a unique surface S', cross-congruent to S.

THEOREM 18. *A necessary and sufficient condition that there exist a surface S' cross-congruent to a given surface S by a non-parabolic map is that there exist a conjugate system on S such that, when it is parametric, the ratio R_1/R_2 of the normal curvatures in its directions satisfy conditions* (40). *Corresponding to each such conjugate system on S there exists a unique surface S':*

$$(41)\qquad x' = \int \frac{R_2}{R_1}x_u du + \frac{R_1}{R_2}x_v dv.$$

If S is a translation surface referred to its generators, the Christoffel symbols in (40) are zero and the equations are satisfied if and only if R_1/R_2 is constant, not unity. Then S', as defined by (41), is a translation surface associate to S.*

THEOREM 19. *There exists a surface S' cross-congruent to a non-minimal translation surface S by a parallel map in which the generators on S are basic if and only if the ratio of the normal curvatures of S in the directions of the generators is constant, not unity. Then S' is a translation surface associate to S.*

COROLLARY. *It is only in this case and that of two associate minimal surfaces that the invariant of the map of two cross-congruent surfaces is constant.*

For, in the minimal case the invariant is known to be constant; it is, in particular, -1 only when the surfaces are adjoints.† In the present case, I is constant, by (36), only if R_1/R_2 is constant, and hence, by (39), only when the two Christoffel symbols vanish; moreover, $I = -1$, only if $R_1/R_2 = \pm i$, and then the two translation surfaces are adjoints.

* Cf. Reference A, p. 313.

† Cf. Reference A, p. 311.

THEOREM 20. *Two associate surfaces of Bianchi* are cross-congruent if and only if they are adjoint minimal surfaces or adjoint translation surfaces with conjugate-imaginary generators such that the ratio of the normal curvatures of either in the directions of the generators is* $\pm i$.

From (40) follows

$$\frac{\partial^2}{\partial u \partial v} \log \frac{R_1}{R_2} = \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}. \tag{42}$$

Since

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = I \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}', \qquad \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = \frac{1}{I} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \tag{43}$$

where the Christoffel symbols with primes refer to S', equations (39) can be put into the forms:

$$\frac{\partial \log R_1/R_2}{\partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}', \quad \frac{\partial \log R_2/R_1}{\partial u} = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'. \tag{44}$$

From (42) and (44) we deduce the relations,

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \qquad \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}',$$

which, taken with (43), yield the equations,

$$h = k', \qquad k = h',$$

where

$$h = \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}, \qquad k = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}$$

are the point invariants of the basic conjugate system on S and h', k' have the same meanings for S'.

THEOREM 21. *The point invariants of the basic conjugate systems of two non-minimal surfaces cross-congruent by a non-parabolic map are equal crosswise. If the basic conjugate system on the one surface has equal point invariants, so has that on the other; this case occurs if and only if the ratio of the normal curvatures in the basic conjugate directions is of the form* $U(u)/V(v)$.

9. **Two examples.**—Given the translation surface S:

$$x_1 = u, \qquad x_2 = v, \qquad x_3 = U(u) + V(v);$$

to determine U, V, so that the ratio of the normal curvatures in the directions of the generators is constant, $\neq \pm 1$ (cf. Th. 19).

Computation gives:

$$\frac{D}{R_1} = \frac{U''}{1 + U'^2}, \qquad \frac{D}{R_2} = \frac{V''}{1 + V'^2}.$$

* These are characterized by the value -1 for I; cf. Reference A, p. 308.

Evidently both ratios must be constant. Setting them equal respectively to $-1/c_1$ and $-1/c_2$, where $c_1^2 \neq c_2^2$, we find that, for a suitable choice of axes,

$$U = c_1 \log \cos \frac{u}{c_1}, \qquad V = c_2 \log \cos \frac{v}{c_2}.$$

Thus

$$S: \qquad x_1 = u, \qquad x_2 = v, \qquad x_3 = c_1 \log \cos \frac{u}{c_1} + c_2 \log \cos \frac{v}{c_2},$$

and, by (41),

$$S': \qquad x_1' = \frac{c_2}{c_1} u, \qquad x_2' = \frac{c_1}{c_2} v, \qquad x_3' = c_2 \log \cos \frac{u}{c_1} + c_1 \log \cos \frac{v}{c_2}.$$

But S' is merely S rotated through a right angle about the x_3-axis. We have here a case of two cross-congruent surfaces which are actually congruent.

If we take as S the surface

$x_1 = u + v, \qquad x_2 = u \tan \alpha + v \tan \beta, \qquad x_3 = U(u) \sec \alpha + V(v) \sec \beta,$

where α and β are constants and $\alpha - \beta \neq n\pi$, we find that U and V must have the same values as in the first example, and that then

$$\frac{R_1}{R_2} = \frac{c_1 \sec \alpha}{c_2 \sec \beta}, \qquad c_1^2 \sec^2 \alpha \neq c_2^2 \sec^2 \beta.$$

The point coördinates of S and S' are readily obtained and it can then be shown that, if $\alpha - \beta \neq (2n + 1)\pi/2$, we have at hand two cross-congruent surfaces which are not congruent.

10. **Parabolic maps.**—Let the surfaces S and S' correspond by a parabolic parallel map. Let the corresponding families of asymptotic lines be the curves $v = \text{const.}$ and choose any other corresponding families of real curves as the curves $u = \text{const.}$ Then scalar functions, $\lambda(u, v)$, $\nu(u, v)$, exist so that*

$$x_u' = \lambda x_u, \qquad x_v' = \nu x_u + \lambda x_v. \tag{45}$$

A necessary condition that the correspondence between S and S' be cross-congruent is that the map be equiareal. But then S and S' are ruled surfaces whose rulings correspond and have the Combescurian ratio, λ, equal to ± 1.† Since S is ruled, with the curves $v = \text{const.}$ as the rulings, it can be represented by the equations,

$$x = \xi(v) + u\eta(v), \tag{46}$$

* Cf. Reference A, p. 303.

† Ibid., p. 316.

where $\xi = \xi(v)$ is the directrix curve referred to its arc v, $\eta(v)$ is the oriented direction of the ruling: $(\eta|\eta) = 1$,* and u is the algebraic distance along the ruling measured from the directrix curve. Let $\alpha = \xi'(v)$ be the oriented direction of the tangent to $\xi = \xi(v)$ and take $\xi = \xi(v)$, in particular, as a secondary asymptotic line: $(\eta\alpha\alpha') = 0$. Then

$$E = 1, \qquad F = (\alpha|\eta) = \cos\theta, \qquad e = 0, \qquad f = \frac{1}{D}(\eta\alpha\eta'),$$

$$g = \frac{1}{D}\{[(\eta\alpha\eta'') + (\eta\eta'\alpha')]u + (\eta\eta'\eta'')u^2\},$$

where θ is the angle under which the ruling intersects the directrix curve.

The surface S' is defined, in terms of S, by (45), where $\lambda = \pm 1$. Since $x'_{uv} = x'_{vu}$, it follows that $\nu_u = 0$. Hence ν is a function of v alone: $\nu = \nu(v)$. We have, then, for S':

$$E' = E, \qquad F' = \lambda\nu E + F, \qquad G' = \nu^2 E + 2\lambda\nu F + G, \qquad D'^2 = D^2,$$
$$e' = 0, \qquad f' = \lambda f, \qquad g' = \nu f + \lambda g, \qquad d'^2 = d^2.$$

Evidently, the total curvatures of S and S' are equal. The mean curvatures are also, if and only if

$$(\lambda - 1)(g - 2Ff) = \nu f.$$

If $\lambda = +1$, $\nu = 0$ and the surfaces are congruent by the map. Hence, for cross-congruence, $\lambda = -1$ and the condition becomes

$$4F - \nu = \frac{2g}{f}.$$

Since $4F - \nu$ depends only on v, so also must g/f. But this is possible only if $g = 0$. Consequently,

$$\nu = 4F = 4\cos\theta.$$

The finite equations of S' are

$$x' = -x + 4\int \eta\cos\theta dv. \tag{47}$$

For cross-congruence, F cannot vanish, that is, S cannot be minimal.

Finally, the vanishing of g is the condition that the secondary asymptotic lines be the curves $u = \text{const.}$, that is, the curves which cut equal segments from the rulings.

THEOREM 22. *There exists a surface S' cross-congruent to a given surface S by a parabolic map if and only if S is a non-minimal ruled surface whose secondary asymptotic lines cut equal segments from the rulings.*† *Then, if*

* If a: a_1, a_2, a_3 and b: b_1, b_2, b_3 are triples, $(a\,|\,b) = a_1b_1 + a_2b_2 + a_3b_3$.

† For further properties of ruled surfaces of this type, cf. Whittemore, "Ruled surfaces with director planes," *Bull. Amer. Math. Soc.*, Vol. 29 (1923), pp. 21–25; Author, "Note on a certain type of ruled surface," *Bull. Amer. Math. Soc.*, Vol. 29 (1923), pp. 341–344.

S is represented by (46), *where the asymptotic lines are the parametric curves, v is the arc of the directrix curve and u the distance along the ruling from it, S' is represented by* (47), *where θ is the angle under which the ruling of S intersects the directrix curve.*

It can readily be shown that $\cos vdv$ is independent of which secondary asymptotic line of S is chosen as the directrix curve. Hence S' is unique. That it is of the same type as S is obvious.

COROLLARY. *The surface S' is unique and is of the same type as the surface S.*

11. **An alternative formulation.**—Let $S: x = x(u, v)$ and $S': x' = x'(u, v)$ be two non-minimal surfaces in one-to-one correspondence by parallel normals. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be the differential coefficients of the common spherical representation and e, f, g and e', f', g' the differential coefficients of the second order of S and S' respectively. Set

$$e' - e = \bar{e}, \qquad f' - f = \bar{f}, \qquad g' - g = \bar{g}. \tag{48}$$

Since the Codazzi equations are linear, homogeneous, differential equations in e, f, g, with coefficients depending only on $\mathcal{E}, \mathcal{F}, \mathcal{G}$, and since e, f, g and e', f', g' satisfy them, so also do $\bar{e}, \bar{f}, \bar{g}$. Consequently, $\bar{e}, \bar{f}, \bar{g}$, taken with $\mathcal{E}, \mathcal{F}, \mathcal{G}$, determine a surface $\overline{S}$ whose position can be so chosen that it is mapped by parallel normals on S and S' and has the parametric representation, $\bar{x} = x' - x$.

Corresponding curvatures of S and S' are equal, by virtue of (48), if and only if

$$e\bar{g} - 2f\bar{f} + g\bar{e} + \bar{e}\bar{g} - \bar{f}^2 = 0, \qquad \mathcal{E}\bar{g} - 2\mathcal{F}\bar{f} + \mathcal{G}\bar{e} = 0. \tag{49}$$

When these conditions are satisfied, S and S' are cross-congruent, unless $\bar{e} = \bar{f} = \bar{g} = 0$. But the existence of $\overline{S}$ rests on the assumption that $\bar{e}, \bar{f}, \bar{g}$ are not all zero.

THEOREM 23. *There exists a surface S' cross-congruent to a given non-minimal surface $S: x = x(u, v)$ by a parallel map if and only if there exists a minimal surface $\overline{S}: \bar{x} = \bar{x}(u, v)$ mapped by parallel normals on S so that the first equation of* (49) *obtains. Then, for each surface $\overline{S}$, there exists a unique surface S', whose position can be so chosen that $x' = x + \bar{x}$.*

A geometrical interpretation of the first condition in (49) can be given in terms of the invariant I of the map of S on $\overline{S}$ and the total curvatures K and $\overline{K}$ of S and $\overline{S}$. We state the result without proof.

THEOREM 24. *There exists a surface S' cross-congruent to a given non-minimal surface S if and only if there exists a minimal surface $\overline{S}$ mapped by parallel normals on S so that*

$$\left(\frac{I+1}{I}\right)^2 = \frac{K}{\overline{K}}.$$

Then S' has the representation $x' = x + \bar{x}$ or $x' = x - \bar{x}$.

The second case reduces to the first by reflecting $\overline{S}$ in the origin. In the first case: $x' = x + \bar{x}$, the invariant I' of the map of S on S' is the square of the invariant I of the map of S on $\overline{S}$: $I' = I^2$. If I' is to be -1, $I = \pm i$ and $K/\overline{K} = 2$.

COROLLARY. *The surfaces S and S' are, in particular, cross-congruent associate surfaces of Bianchi, if and only if the invariant of the map of S on $\overline{S}$ is $\pm i$ and the total curvature of S is double that of $\overline{S}$.*

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